

Equality is coalgebraic

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joint work with Jacopo Emmenegger and Fabio Pasquali

May 20, 2022

Categories and Logic

Logic for categories monoidal closed category
cartesian closed category
Grothendieck topos
exact category
elementary topos
pretopos
Heyting pretopos
arithmetic universe
logos...

Category theory for logic hyperdoctrine
existential elementary doctrine
tripos
 $D: \mathcal{C}^{\text{op}} \longrightarrow \mathcal{P}os$ indexed poset
Grothendieck fibration
elementary fibration...

F.W. Lawvere

Adjointness in foundations *Dialectica* 1969, available also in *Repr. TAC*

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Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$ is a *primary doctrine* when

▶ \mathcal{B} has finite products

▶ each fibre $P(b)$ has finite meets $P(b) \times P(b) \begin{array}{c} \xleftarrow{\langle \text{id}, \text{id} \rangle} \\ \perp \\ \xrightarrow{\wedge_b} \end{array} P(b) \begin{array}{c} \xrightarrow{!} \\ \perp \\ \xleftarrow{\top_b} \end{array} 1$

▶ the families $(\wedge_b)_b: P \times P \dashrightarrow P$ and $(\top_b)_b: 1 \dashrightarrow P$ are natural

F.W. Lawvere

Equality in hyperdoctrines and the comprehension schema as an adjoint functor

Proc.AMS Symp.Pure Math. 1970

B. Jacobs

Categorical Logic and Type Theory 1999

A.M. Pitts

Categorical logic *Handbook of Logic in Computer Science, vol. 5, 2000*

M.E. Maietti, G. R.

Quotient completion for the foundation of constructive mathematics *Log.Univer.* 2013

$$\wp: \mathit{Set}^{\text{op}} \longrightarrow \mathit{Pos}$$

The powerset functor

$$\begin{array}{ccc} \mathit{Set}^{\text{op}} & \xrightarrow{\wp} & \mathit{Pos} \\ S & \longmapsto & \wp(S) \\ \downarrow f & \longmapsto & \downarrow f^{-1} \\ S' & \longmapsto & \wp(S') \end{array}$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \quad f^{-1}(S) = S'$$

$$\text{LT}^{\mathcal{T}}: \text{Ctx}_{\mathcal{L}}^{\text{op}} \longrightarrow \text{Pos}$$

\mathcal{T} : a theory in the \wedge -fragment of first order logic

$$\text{Ctx}_{\mathcal{L}}: (x_{i_1}, \dots, x_{i_n}) \xrightarrow{(t_1, \dots, t_m)} (x_{j_1}, \dots, x_{j_m}) \xrightarrow{(s_1, \dots, s_k)} (x_{l_1}, \dots, x_{l_k})$$

$$(s_1[t_1/x_{j_1}, \dots, t_m/x_{j_m}], \dots, s_k[t_1/x_{j_1}, \dots, t_m/x_{j_m}])$$

$\text{LT}^{\mathcal{T}}(x_{i_1}, \dots, x_{i_n})$: the Lindenbaum-Tarski algebra of formulas with free variables among $(x_{i_1}, \dots, x_{i_n})$

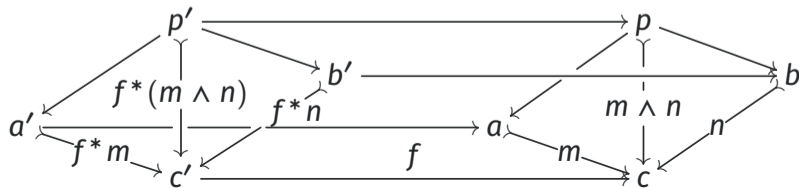
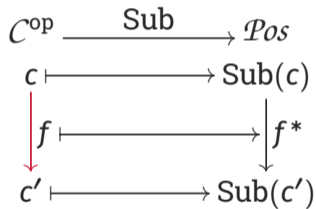
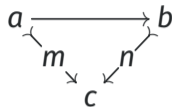
$$\begin{array}{ccc} \text{Ctx}_{\mathcal{L}}^{\text{op}} & \xrightarrow{\text{LT}^{\mathcal{T}}} & \text{Pos} \\ \vec{x}_j \vdash & \longrightarrow & \text{LT}^{\mathcal{T}}(\vec{x}_j) \\ \downarrow \vec{t} & \longrightarrow & \downarrow (-)[\vec{t}/\vec{x}_j] \\ \vec{x}_i \vdash & \longrightarrow & \text{LT}^{\mathcal{T}}(\vec{x}_i) \end{array}$$

$$(\varphi \wedge \psi)[\vec{t}/\vec{x}_i] = \varphi[\vec{t}/\vec{x}_i] \wedge \psi[\vec{t}/\vec{x}_i] \quad \top[\vec{t}/\vec{x}_i] = \top$$

Sub: $\mathcal{C}^{\text{op}} \longrightarrow \mathcal{P}os$

\mathcal{C} : a category with finite limits

Sub(c): the poset reflection of the preorder



$$\text{Sub}_{\mathcal{M}}: \mathcal{C}^{\text{op}} \longrightarrow \mathcal{P}os$$

\mathcal{C} : a category with finite limits

\mathcal{M} : a class of monos in \mathcal{C} closed under identities and pullbacks

$\text{Sub}_{\mathcal{M}}(c) \subseteq \text{Sub}(c)$: the sub-poset of $\text{Sub}(c)$ on those equivalence classes with representatives in \mathcal{M}

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & \xrightarrow{\text{Sub}_{\mathcal{M}}} & \mathcal{P}os \\
 c \vdash & \longrightarrow & \text{Sub}_{\mathcal{M}}(c) \\
 \downarrow f & \longrightarrow & \downarrow f^* \\
 c' \vdash & \longrightarrow & \text{Sub}_{\mathcal{M}}(c')
 \end{array}$$

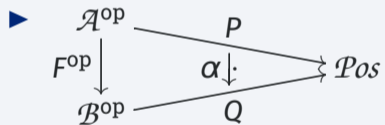
$\text{Sub}_{\mathcal{M}}: \mathcal{C}^{\text{op}} \longrightarrow \mathcal{P}os$ is a *primary sub-doctrine* of $\text{Sub}: \mathcal{C}^{\text{op}} \longrightarrow \mathcal{P}os$

Homomorphisms of primary doctrines

Definition

A *homomorphism of primary doctrines* from $P: \mathcal{A} \longrightarrow \mathcal{P}os$ to $Q: \mathcal{B} \longrightarrow \mathcal{P}os$ is a pair (F, α) where

▶ $F: \mathcal{A} \longrightarrow \mathcal{B}$ preserves finite products



▶ every $\alpha_a: P(a) \longrightarrow Q(F(a))$ preserves finite meets

They actually form a 2-category $\mathcal{P}Doct$

Homomorphisms of primary doctrines

Examples

$$\begin{array}{ccc} C^{\text{op}} & \xrightarrow{\text{Sub}_{\mathcal{M}}} & \mathcal{P}os \\ \text{Id}_C^{\text{op}} \downarrow & \text{In} \downarrow & \wr \\ C^{\text{op}} & \xrightarrow{\text{Sub}} & \end{array}$$

A homomorphism

$$\begin{array}{ccc} \text{Ctx}_L^{\text{op}} & \xrightarrow{\text{LT}^{\mathcal{T}}} & \mathcal{P}os \\ F^{\text{op}} \downarrow & \alpha \downarrow & \wr \\ \text{Set}^{\text{op}} & \xrightarrow{\emptyset} & \end{array}$$

is exactly a model of the Horn sentences provable in \mathcal{T}

$$H^{(-)}: \mathit{Set}^{\text{op}} \longrightarrow \mathit{Pos}$$

H : an inf-semilattice

$$\mathit{Set}^{\text{op}} \xrightarrow{H^{(-)}} \mathit{Pos}$$

$$S \longmapsto H^S \quad \text{with the pointwise order}$$

$$f \downarrow \longmapsto \downarrow - \circ f$$

$$S' \longmapsto H^{S'}$$

$$(H^f(\varphi \wedge \psi))(s) = (\varphi \wedge \psi)(f(s)) = \varphi(f(s)) \wedge \psi(f(s))$$

$$= H^f(\varphi)(s) \wedge H^f(\psi)(s) = (H^f(\varphi) \wedge H^f(\psi))(s)$$

$$\mathbf{Vrn}: \mathcal{A}^{\text{op}} \longrightarrow \mathcal{P}os$$

\mathcal{A} : a category with finite (strict) products and weak pullbacks

$\mathbf{Vrn}(a)$: the poset reflection of the category \mathcal{A}/a

$$\begin{array}{ccc} v & \xrightarrow{\quad} & w \\ & \searrow e & \swarrow d \\ & & a \end{array}$$

$$\mathcal{A}^{\text{op}} \xrightarrow{\mathbf{Vrn}} \mathcal{P}os$$

$$a \longmapsto \mathbf{Vrn}(a) \ni [e]$$

$$\begin{array}{ccc} \downarrow f & \longmapsto & \downarrow \mathbf{Vrn}(f) \\ a' & \longmapsto & \mathbf{Vrn}(a') \ni [e'] \end{array}$$

where

$$\begin{array}{ccc} v' & \xrightarrow{\quad} & v \\ \downarrow e' & \text{w.p.} & \downarrow e \\ a' & \xrightarrow{f} & a \end{array}$$

$$\mathbf{Vrn}(a) \xleftarrow{f \circ -} \mathbf{Vrn}(a') \xrightarrow{\mathbf{Vrn}(f)} \mathbf{Vrn}(a)$$

The internal logic of a primary doctrine

Given a primary doctrine $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$

consider the many-sorted language the objects of \mathcal{B} as sorts

Write arrows in \mathcal{B} as terms in context $x_1 : b_1, \dots, x_n : b_n$ as follows:

- ▶ $x_i : b_i$ is the i -th projection $\pi_i : b_1 \times \dots \times b_n \rightarrow b_i$
- ▶ for $t : b$ and $f : b \rightarrow b'$ in \mathcal{B} , the term $f(t)$ is the composite

$$b_1 \times \dots \times b_n \xrightarrow{t} b \xrightarrow{f} b'$$

- ▶ for $t_1 : b'_1 \dots t_m : b'_m$ the term $\langle t_1, \dots, t_m \rangle : b'_1 \times \dots \times b'_m$ is the unique arrow determined into the product $b'_1 \times \dots \times b'_m$ by the family $t_j : b_1 \times \dots \times b_n \rightarrow b'_j \quad j = 1, \dots, m$

Terms $\vec{x} \mid t_1 : b$ and $\vec{x} \mid t_2 : b$ are equal iff their equality follows from the axioms

$$\text{id}_b(t) = t \quad g(f(t)) = (g \circ f)(t) \quad \pi_i \langle t_1, \dots, t_n \rangle = t_i \quad \langle \pi_1(t), \dots, \pi_n(t) \rangle = t$$

The internal logic of a primary doctrine, II

Write objects in fibres $P(b)$ as formulas in context $x_1 : b_1, \dots, x_n : b_n$ as follows:

- for $t : b$ and $\varphi \in P(b)$ in \mathcal{B} , the formula $\varphi(t)$ is $P(t)(\varphi) \in P(b_1 \times \dots \times b_n)$

Write $x_1 : b_1, \dots, x_n : b_n \mid \Gamma \vdash_{\mathcal{T}_p} \varphi$ when $\mathbb{M}_{\gamma \in \Gamma} \gamma \leq \varphi$ in $P(b_1 \times \dots \times b_n)$

The following rules are derivable

$$\begin{array}{c}
 \frac{\bar{x} \mid \Gamma, \varphi, \psi, \Delta \vdash_{\mathcal{T}_p} \theta}{\bar{x} \mid \Gamma, \psi, \varphi, \Delta \vdash_{\mathcal{T}_p} \theta} \quad \frac{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \theta}{\bar{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_p} \theta} \\
 \\
 \frac{}{\bar{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_p} \varphi} \quad \frac{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \varphi \quad \bar{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_p} \psi}{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \psi} \quad \frac{\bar{x} \mid \Gamma, \varphi \vdash_{\mathcal{T}_p} \psi}{\bar{x}' \mid \Gamma, \varphi(t) \vdash_{\mathcal{T}_p} \psi(t)} \\
 \\
 \frac{}{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \top} \quad \frac{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \varphi \wedge \psi}{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \varphi} \quad \frac{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \varphi \wedge \psi}{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \psi} \quad \frac{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \varphi \quad \bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \psi}{\bar{x} \mid \Gamma \vdash_{\mathcal{T}_p} \varphi \wedge \psi}
 \end{array}$$

Elementary doctrines

Definition

A primary doctrine $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$ is *elementary* when for each parametrized diagonal $d_{z,b} \stackrel{\text{df}}{=} \langle \pi_1, \pi_2, \pi_2 \rangle : z \times b \longrightarrow z \times b \times b$ in \mathcal{B}

$$P(z \times b \times b) \begin{array}{c} \xrightarrow{P(d_{z,b})} \\ \top \\ \xleftarrow{\Xi_{d_{z,b}}} \end{array} P(z \times b)$$

- ▶ the left adjoints are natural in z
- ▶ they satisfy Frobenius Reciprocity

$$\Xi_{d_{z,b}}[P(d_{z,b})(\alpha) \wedge \beta] \xrightarrow{\sim} \alpha \wedge \Xi_{d_{z,b}}(\beta)$$

F.W. Lawvere

Equality in hyperdoctrines and the comprehension schema as an adjoint functor

Proc.AMS Symp.Pure Math. 1970

Examples of elementary doctrines

$$\mathit{Set}^{\text{op}} \xrightarrow{\wp} \mathit{Pos}$$

$$\mathcal{A}^{\text{op}} \xrightarrow{\mathbf{Vrn}} \mathit{Pos} \quad \text{for } \mathcal{A} \text{ with finite products and weak pullbacks}$$

$$\mathcal{C}^{\text{op}} \xrightarrow{\mathbf{Sub}} \mathit{Pos} \quad \text{for } \mathcal{C} \text{ with finite products and pullbacks of monos}$$

$$\mathcal{C}^{\text{op}} \xrightarrow{\mathbf{Sub}_{\mathcal{M}}} \mathit{Pos} \quad \text{for } \mathcal{C} \text{ with finite products and } \mathcal{M} \text{ a class of monos closed under pullbacks and diagonals}$$

A characterization of elementary doctrines

Theorem

Let $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$ be a primary doctrine. TFAE

- (i) P is elementary
- (ii) for every b in \mathcal{B} there is δ_b^P in $P(b \times b)$ such that
 - (a) $x : b \vdash_{\mathcal{T}_P} \delta_b^P(x, x)$
 - (b) $x_1, y_1 : b_1, x_2, y_2 : b_2 \mid \delta_{b_1}^P(x_1, y_1), \delta_{b_2}^P(x_2, y_2) \vdash_{\mathcal{T}_P} \delta_{b_1 \times b_2}^P(\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle)$
 - (c) $x, y : b \mid \beta(x), \delta_b^P(x, y) \vdash_{\mathcal{T}_P} \beta(y)$ for every β in $P(b)$

Corollary

Let $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$ be an elementary doctrine. Then

- ▶ $x, y : b, z : b' \mid \alpha(x, z), \delta_b^P(x, y) \vdash_{\mathcal{T}_P} \alpha(y, z)$ for every α in $P(b \times b')$
- ▶ $x, y : b \mid \delta_b^P(x, y) \vdash_{\mathcal{T}_P} \delta_{b'}^P(f(x), f(y))$

More examples of elementary doctrines

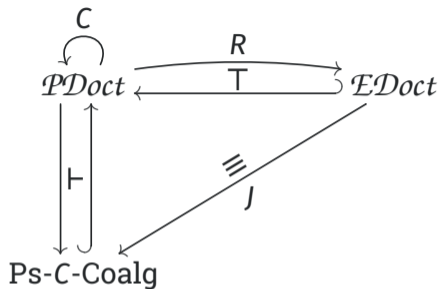
$Set^{op} \xrightarrow{H^{(-)}} Pos$ for H an inf-semilattice with a least element

$$S \times S \xrightarrow{\delta_S^P} H$$
$$(s_1, s_2) \longmapsto \begin{cases} \top & \text{if } s_1 = s_2 \\ \perp & \text{if } s_1 \neq s_2 \end{cases}$$

$Ctx_{\mathcal{L}}^{op} \xrightarrow{LT^T} Pos$ for T a theory in the $\wedge =$ -fragment of first order logic

$\delta_{(x_{i_1}, \dots, x_{i_n})}^{LT^T}$ is the formula $x_{i_1} = x'_{i_1} \wedge \dots \wedge x_{i_n} = x'_{i_n}$

Elementary doctrines are 2-coalgebras



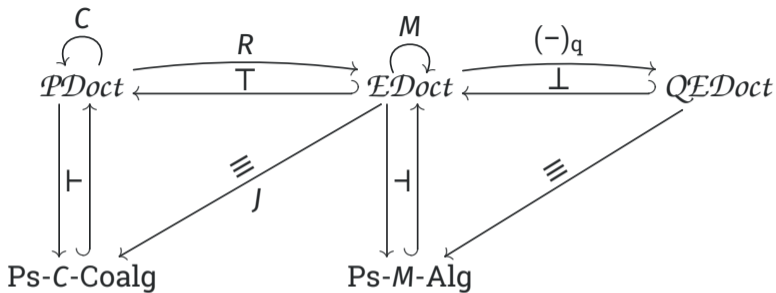
Theorem

The comonad on \mathcal{C} is KZ and the comparison functor J is a 2-equivalence

J. Emmenegger, F. Pasquali, G. R.

Elementary doctrines as coalgebras *J.Pure Appl.Algebra* 2020

Elementary doctrines are 2-coalgebras



M.E. Maietti, G. R.

Elementary quotient completion *Theory Appl.Categ.* 2013

F. Pasquali

A co-free construction for elementary doctrines *Appl.Categ. Structures* 2015

D. Trotta

Existential completion and pseudo-distributive laws:
an algebraic approach to the completion of doctrines *PhD Thesis* 2019

The functor of the comonad on $\mathcal{P}\mathcal{D}o\mathcal{c}t$

Given $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}o\mathcal{s}$ a primary doctrine

the primary doctrine $\mathcal{C}(P)$ is $\mathcal{E}_P^{\text{op}} \xrightarrow{\text{Des}_P} \mathcal{P}o\mathcal{s}$ where

Objects of \mathcal{E}_P are (b, ρ) where

- ▶ b is an object of \mathcal{B}
- ▶ ρ is an object of $P(b \times b)$ such that
 - ▶ $x : b \vdash_{\mathcal{T}_P} \rho(x, x)$
 - ▶ $x, y : b \mid \rho(x, y) \vdash_{\mathcal{T}_P} \rho(y, x)$
 - ▶ $x, y, z : b \mid \rho(x, y), \rho(y, z) \vdash_{\mathcal{T}_P} \rho(x, z)$

Arrows of \mathcal{E}_P are $f : (b, \rho) \longrightarrow (b', \rho')$ where

- ▶ $f : b \longrightarrow b'$ is in \mathcal{C} such that
 - ▶ $x, y : b \mid \rho(x, y) \vdash_{\mathcal{T}_P} \rho'(f(x), f(y))$

$\text{Des}_P(b, \rho)$ is the sub-poset of $P(b)$ on the objects β in $P(b)$ such that

$$x, y : b \mid \beta(x), \rho(x, y) \vdash_{\mathcal{T}_P} \beta(y)$$

Examples

$$\mathcal{P}Doct \xrightleftharpoons[\mathcal{T}]{\mathcal{R}} \mathcal{E}Doct$$

$$Ct\chi_T^{\text{op}} \xrightarrow{\text{LT}_T} Pos$$

$$Ct\chi_{T^{\text{eq}}}^{\text{op}} \xrightarrow{\text{LT}_{T^{\text{eq}}}} Pos$$

$$\mathcal{A}^{\text{op}} \xrightarrow{\text{Vrn}} Pos$$

$$\mathcal{E}_{\text{Vrn}}^{\text{op}} \xrightarrow{\quad} (\mathcal{A}_{\text{ex}})^{\text{op}} \xrightarrow{\text{Sub}} Pos$$

$R(\text{Vrn})$

B. Poizat

Une théorie de Galois imaginaire *J. Symb. Log.* 1983

A. Carboni, E. Vitale

Regular and exact completions *J. Pure Appl. Algebra* 1998

Examples

$$\mathcal{P}Doct \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{T} \end{array} \mathcal{E}Doct$$

$$\mathbf{Fzs}(H)^{\text{op}} \xrightarrow{\text{Gr}(H^{(-)})} \mathcal{P}os \quad \mathcal{E}_{\text{Gr}(H^{(-)})}^{\text{op}} \begin{array}{c} \xrightarrow{\text{sep}([\mathcal{H}^{\text{op}}, \text{Set}]^{\text{op}})} \\ \xrightarrow{R(\text{Gr}(H^{(-)}))} \end{array} \xrightarrow{\text{ClSub}} \mathcal{P}os$$

M.P. Fourman, D.S. Scott

Sheaves and logic *Applications of sheaves* LNM#753, 1979

G.P. Monro

Quasitopoi, logic and Heyting valued-models *J.Pure Appl.Algebra* 1986

O. Wyler

Lecture notes on topoi and quasitopoi 1991

Examples

$$\mathcal{P}Doct \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{T} \end{array} \mathcal{E}Doct$$

$$\text{Fzs}(H)^{\text{op}} \xrightarrow{\text{Gr}(H^{(-)})} \mathcal{P}os \quad \mathcal{E}_{\text{Gr}(H^{(-)})}^{\text{op}} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{R(\text{Gr}(H^{(-)}))} \end{array} \text{sep}([\mathcal{H}^{\text{op}}, \text{Set}]^{\text{op}}) \xrightarrow{\text{ClSub}} \mathcal{P}os$$

Proposition

The primary doctrine $\text{Fzs}(H)^{\text{op}} \xrightarrow{\text{Gr}(H^{(-)})} \mathcal{P}os$ is the comprehension completion of the primary doctrine $\text{Set}^{\text{op}} \xrightarrow{H^{(-)}} \mathcal{P}os$

Examples

$$\mathcal{P}\mathcal{D}o\mathcal{c}t \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{T} \end{array} \mathcal{E}\mathcal{D}o\mathcal{c}t$$

$$\text{Fzs}(H)^{\text{op}} \xrightarrow{\text{Gr}(H^{(-)})} \mathcal{P}os \quad \mathcal{E}_{\text{Gr}(H^{(-)})}^{\text{op}} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \text{sep}([\mathcal{H}^{\text{op}}, \text{Set}]^{\text{op}}) \xrightarrow{\text{ClSub}} \mathcal{P}os$$

$$R(\text{Gr}(H^{(-)}))$$

$$\text{Fzs}(H): \quad (S, \alpha) \xrightarrow[f: S \rightarrow S']{(S', \alpha')}$$

s.t. $\alpha \in H^S$ s.t. $\alpha(s) \leq \alpha'(f(s)), s \in S$ s.t. $\alpha' \in H^{S'}$

$\text{Gr}(H^{(-)})(S, \alpha)$: the sub-poset of H^S on the objects β such that $\beta(s) \leq \alpha(s)$ for all $s \in S$

Explaining the quotients

The quotients $\mathcal{E}_{\text{Vrn}} \longrightarrow \mathcal{A}_{\text{ex}}$ and $\mathcal{E}_{\text{Gr}(H(-))} \longrightarrow \text{sep}([\mathcal{H}^{\text{op}}, \text{Set}])$ identify arrows exactly when they have the same actions on “parametrized propositions”

Proposition

Let $\mathcal{B}^{\text{op}} \xrightarrow{P} \mathcal{P}os$ be an elementary doctrine.

For $f, g: b \rightarrow c$ the following are equivalent

- (i) $x : b, z : b' \mid \alpha(f(x), z) \dashv\vdash_{\mathcal{T}_p} \alpha(g(x), z)$ for every α in $P(c \times b')$
- (ii) $x : b \vdash_{\mathcal{T}_p} \delta_c^P(f(x), g(x))$

M.E. Maietti, G. R.

Unifying Exact Completions *Appl.Categ. Structures* 2013

C.J. Cioffo

Homotopy setoids and generalized quotient completion

PhD thesis, 2022

Elementary fibrations

Definition

A fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is *elementary* when

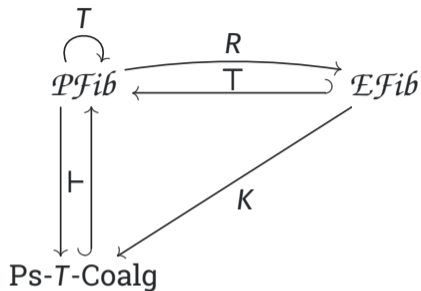
- ▶ \mathcal{B} and \mathcal{E} have finite products, and p preserves them
- ▶ for each parametrized diagonal $d_{z,b} \stackrel{\text{df}}{=} \langle \pi_1, \pi_2, \pi_2 \rangle : z \times b \longrightarrow z \times b \times b$ in \mathcal{B}

$$p^{-1}(z \times b \times b) \begin{array}{c} \xrightarrow{p^{-1}(d_{z,b})} \\ \top \\ \xleftarrow{\Xi_{d_{z,b}}} \end{array} p^{-1}(z \times b)$$

and the left adjoints are natural in z , and satisfy Frobenius Reciprocity

$$\Xi_{d_{z,b}}[p^{-1}(d_{z,b})(\alpha) \wedge \beta] \xrightarrow{\sim} \alpha \wedge \Xi_{d_{z,b}}(\beta)$$

Elementary fibrations are pseudo-2-coalgebras



Theorem

The comonad on T is KZ and the comparison functor K is a 2-equivalence

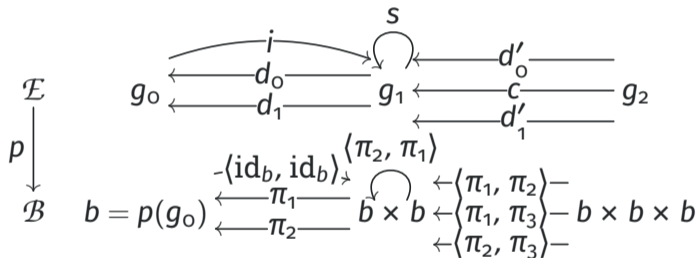
J. Emmenegger, F. Pasquali, G. R.

Equality is pseudo-coalgebraic *in preparation*

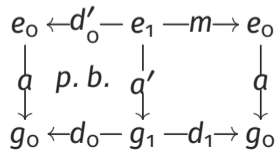
The functor of the comonad on $\mathcal{P}Fib$

For a fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$, \mathcal{B} and \mathcal{E} with finite products, preserved by p
the fibration $T(p)$ is as follows:

\mathcal{E}_p is a full subcategory of $\mathbf{Gpd}(\mathcal{E})$:



\mathcal{D}_p is a full subcategory of $\mathbf{Act}(\mathcal{E})$:





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