Localisable monads: From global to local

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Outline

1. Motivational problems
2. Central idempotents
3. Localisable monads
4. Abstract characterisation of localisable monads
5. Back to motivational problems

Motivational Problems

Monads

- Way of providing objects and morphisms with additional context.

- Used to describe side-effects in programming semantics, e.g. reading and writing from a memory store

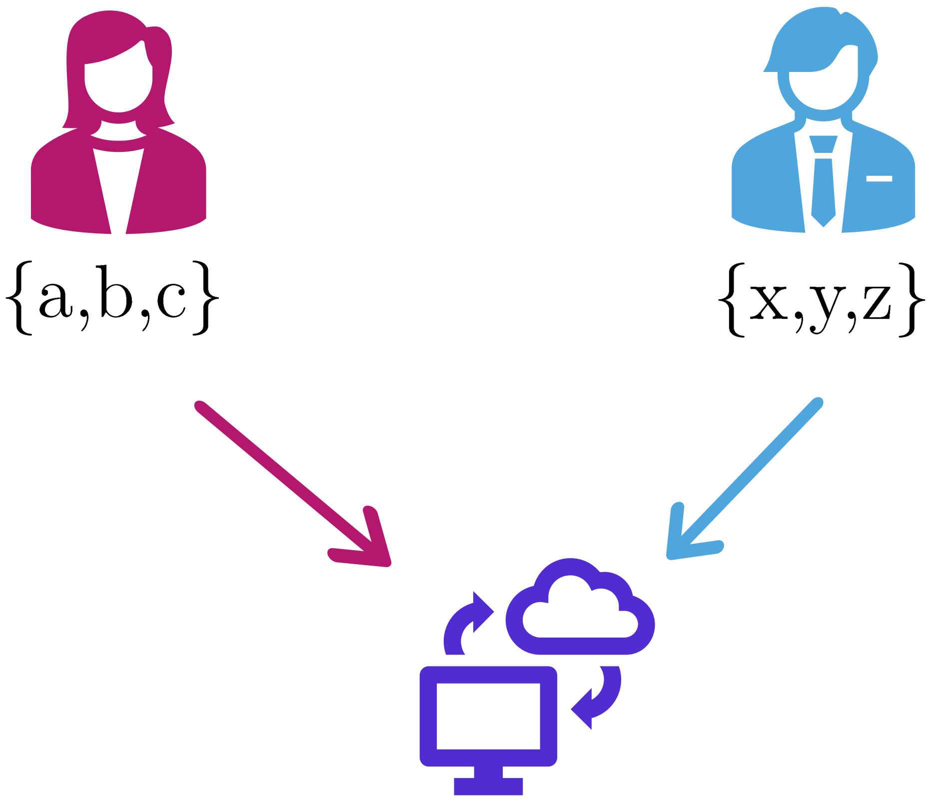
- From this perspective: if we start with a "large" or global monad, can we obtain "smaller" or local monad-like structures?

$$(T, \mu, \eta)$$

Endofunctor

Natural transformations

Example: Concurrency



Want: Category **C** with monad M that "restricts" to $\mathbf{C}_1$ and $\mathbf{C}_2$ with monads M_1 and M_2.

Example: From Set to $\mathbf{Set}^n$

- **Set**:

$$T(-) = S \multimap (- \times S)$$

  - Want something with "decomposable underlying data"

What about something like this?

- $\mathbf{Set}^n$

$$T(A_1, \ldots, A_n) = (S_1, \ldots, S_n) \multimap \big((A_1, \ldots, A_n) \times (S_1, \ldots, S_n)\big)$$

  - Similar question on **Hilb** and $\mathbf{Hilb}^n$

Central Idempotents

Intrinsic Structure

(To a monoidal category)

Intrinsic structure?

- **Idea:** Want a way to split up monads into “smaller monads”

- We want to identify a structure in a monoidal category that will enable this.

  - ➢ **Central idempotents!**

- Let’s look at a monoidal category with interesting “bigger” to “smaller” structures…

M. Boyarchenko and V. Drinfeld. Idempotents in monoidal categories. http://www.math.uchicago.edu/~mitya/idempotents.pdf

Moliner, Pau Enrique, Chris Heunen, and Sean Tull. "Tensor topology." *Journal of Pure and Applied Algebra* (2020)

Motivating example

1 Rings and modules

- Let R be a commutative unital ring.
- Consider the monoidal category of R-modules (and R linear morphisms) ($\otimes$: $\otimes$ of R-modules and $I = R$)
- Idempotents ideals

Ideal, map $u : U \to R$

$$u \otimes U : U \otimes U \longrightarrow U$$
$$x \otimes y \longrightarrow xy$$

Idempotent

$$U = U^2 \longrightarrow U \otimes U$$
$$\sum_i x_i y_i \longrightarrow \sum_i x_i \otimes y_i$$

We have that $u \otimes U$ is invertible.

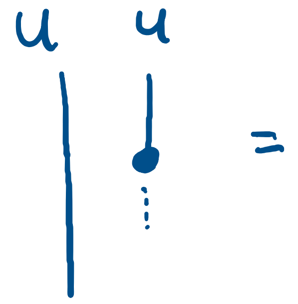
Central idempotents

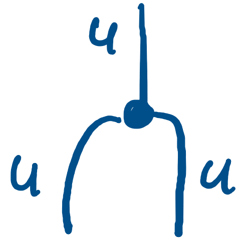
(in a symmetric monoidal category)

Definition

- Morphism $u : U \to I$
- Such that $\rho_U \circ (U \otimes u) = \lambda_U \circ (u \otimes U) : U \otimes U \to U$ is invertible.

Note: I is a central idempotent too!





Equivalence Class

- We identify $u : U \to I$ and $v : V \to I$ when there is an isomorphism $m : U \to V$ such that $u = v \circ m$.

★ Note: u is completely determined by U.

Motivating example

2 Sheaves and opens

- Sheaf $F : \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$
- Monoidal category of sheaves over X
- Subterminal sheaves:

$$\chi_U : \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$$
$$V \mapsto \begin{cases} \{*\} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

- Subterminal object → constant presheaves → **opens of X**

- In a cartesian category, central idempotent are exactly subterminal objects.
- We can think of central idempotents as open subsets of a hidden base space that any symmetric monoidal category comes equipped with.

More examples

3 Set

$(\mathbf{Set}, \times, \{*\})$

Central idempotents: $0, \ 1$

$(\mathbf{Set}^n, \times, (\{*\}, ..., \{*\}))$

Central idempotents: $\{0,1\}^n$

e.g. $(A,B,C) \otimes (1,0,1) \simeq (A,0,C)$

4 Lattice

- Meet semilattice $(L, \wedge, 1)$ as a category
- **Central idempotents:** all elements of L

★ Note: Central idempotents always form a semilattice!

What is the plan?

0. **Want: To use central idempotent to define a notion of local monads**

1. **Central idempotent $\rightarrow$ “bunch” of local categories**

2. **Condition to define monads on these categories**

Categories restricted to central idempotents

1 **We want a "bunch" of categories (for each u)**

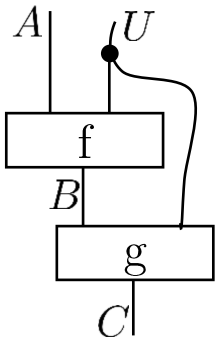
Idea: For each u define a new category $\mathbf{C}||_u$.

Objects: Objects of $\mathbf{C}$

Morphisms: $A \longrightarrow B$ in $\mathbf{C}||_u$
corresponds to $A \otimes U \longrightarrow B$ in $\mathbf{C}$

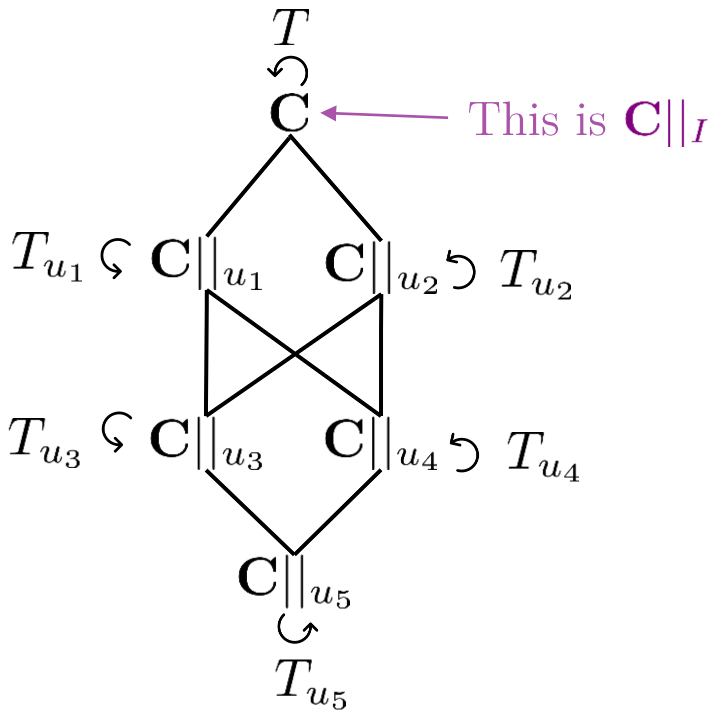
magic
happens
here

Composition:



This is the coKleisli composition
of the comonad $(-) \otimes U$

Identity: $A \otimes u$



2 **Can we put a monad on each new category?**

Moliner, P. E., C. Heunen, and S. Tull. "Tensor topology." *Journal of Pure and Applied Algebra* (2020)

Localisable Monads

(New Stuff!)

2 Can we put a monad on each new category?

a Conditions $\to$ Localisable

b Localisable $\to$ Local

Localisable monads

Definition:

A monad T is *localisable at a central idempotent* U if for any object A there are partial strengths

$$\mathrm{st}_{A,U} : T(A) \otimes U \to T(A \otimes U)$$

(satisfying some compatibility axioms)

★ Localisable if localisable at all U.

Example: Strong monads.

Example: a monad T on a cartesian closed category if

$$T(A \times B) \simeq T(A) \times T(B)$$

Local monads

We want to define LOCAL monads on $\mathbf{C}||_u$

Assumption: Monad T on $\mathbf{C}$.

$$T_u(A) = T(A)$$

$$T_u(f) = ?$$

$$T(A) \otimes U \xrightarrow{\mathrm{st}} T(A \otimes U) \xrightarrow{T(f)} T(B)$$

$$\eta^u_A = \eta_A \otimes u$$

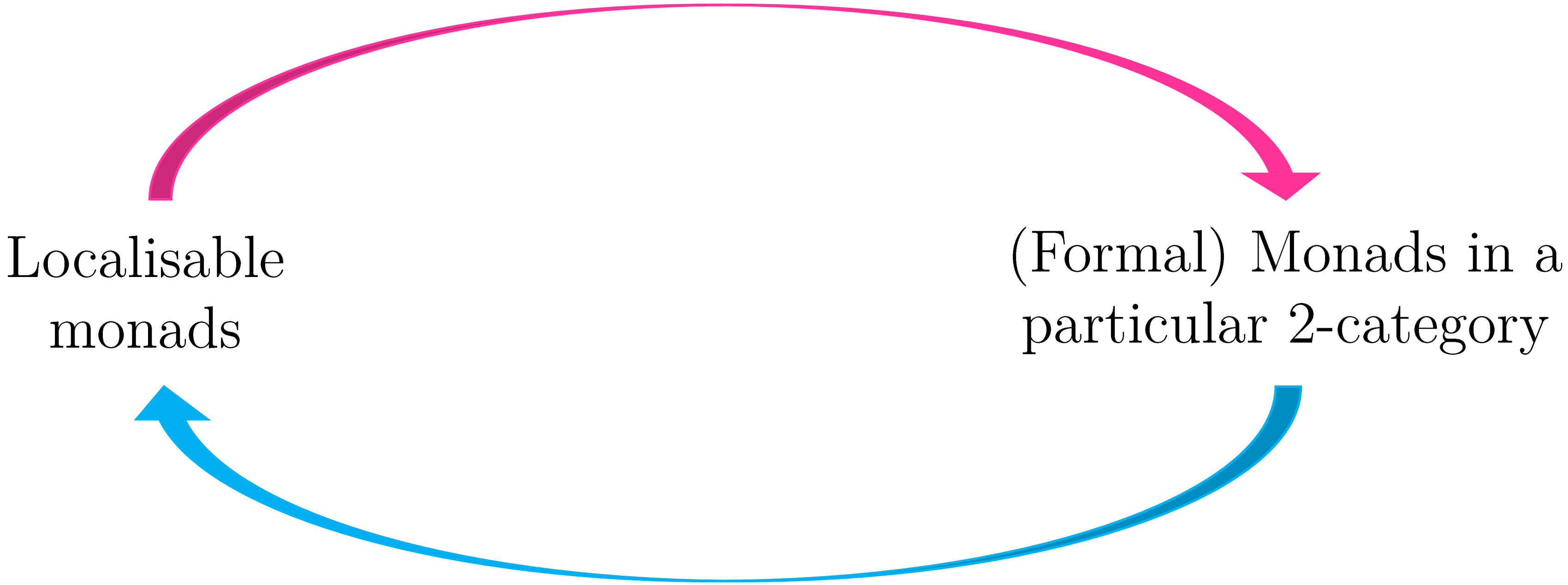
$$\mu^u_A = \mu_A \otimes u$$

Outline

1. Motivational problems
2. Intrinsic structure (Central idempotents)
3. Localisable monads
4. Abstract characterisation of localisable monads
5. Back to motivational problems

Abstract Characterisation

Objective



<u>Theorem:</u> The above are equivalent.

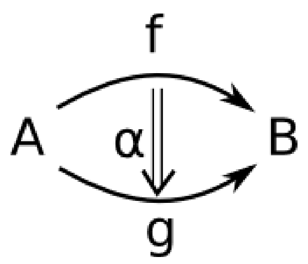
2-categories

Street (1972): Formal theory of monads

Theory of monads in arbitrary 2-categories

2-category:

0-cells
1-cells: maps between 0-cells
2-cells: maps between 1-cells



2-cat

2-category: $K = [\mathbf{ZI(C)^{op}}, \mathbf{Cat}]$

central idempotents

0-cells: 2-functors
1-cells: nat. transf. (i.e. components are 2-functors for each U)
2-cells: modifications (i.e. components are nat. transf. for each U)

Note: This can get confusing!!!

Monads in 2-categories

What is a monad in a 2-category?

i) 1-cell

$T : F \to F$

ii) 2-cell

$\mu : TT \longrightarrow T$

$\eta : 1_F \longrightarrow T$

Satisfying the usual monad laws.

Choose 0-cell $\overline{\mathbf{C}} : \mathbf{ZI(C)^{op}} \longrightarrow \mathbf{Cat}$

$u \longmapsto \mathbf{C}||_u$

$T : \overline{\mathbf{C}} \longrightarrow \overline{\mathbf{C}}$ $\rightarrow T_u : \mathbf{C}||_u \longrightarrow \mathbf{C}||_u$ Functor

1-cell (Nat. Transf)

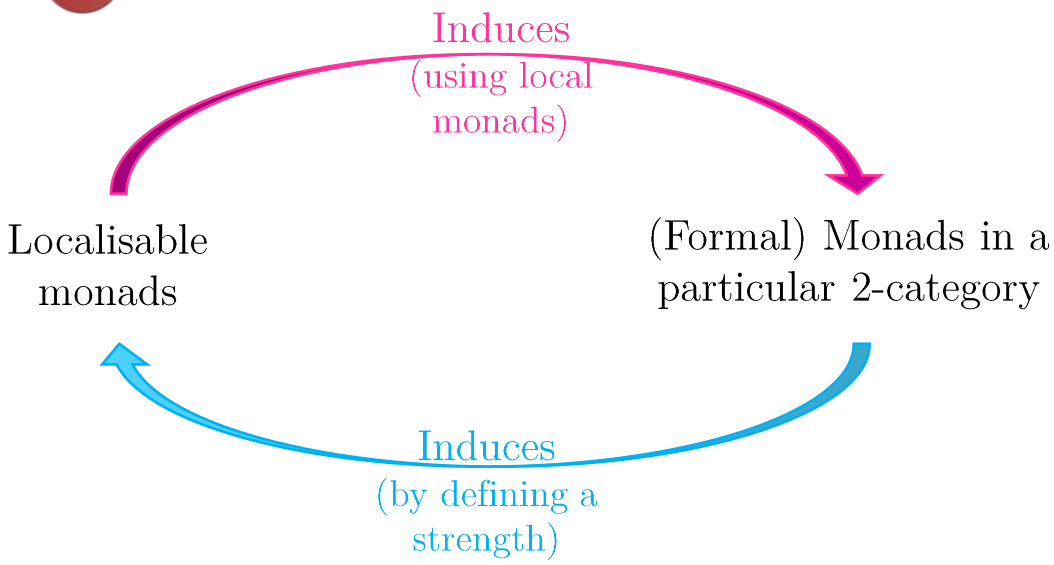
$\mu : TT \longrightarrow T$ $\rightarrow \mu_u : T_u T_u \longrightarrow T_u$

$\eta : 1_{\overline{\mathbf{C}}} \longrightarrow T$ $\rightarrow \eta_u : 1_{\mathbf{C}||_u} \longrightarrow T_u$

Nat. transf.

2-cells (modifications)

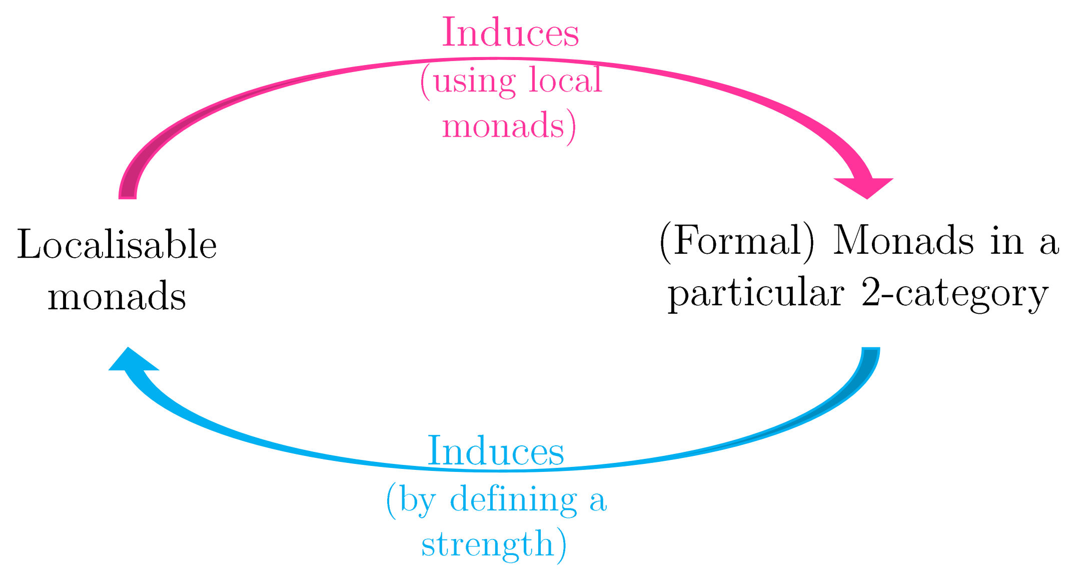
Main Theorem (new result)



Theorem: *For a monoidal category* $\mathbf{C}$ *there is a bijective correspondence between localisable monads on* $\mathbf{C}$ *and formal monads on* $\mathbf{C}$ *in* $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}},\mathbf{Cat}]$.

- Top arrow: Follows by definition
- Bottom arrow: Follows from properties of the strength defined
- Formal $\rightarrow$ Localisable $\rightarrow$ Formal: Naturality of the strength
- Localisable $\rightarrow$ Formal $\rightarrow$ Localisable: The structure of the strength introduced in the bottom arrow allows us to recover the original strength of the localisable monad

What is behind this?



- For each v we want to define a strength

$$\text{st} : T_v(A) \otimes U \longrightarrow T_v(A \otimes U)$$

- $\mathbf{C}||_u$ is the coKleisli category of the comonad $- \otimes U$ on $\mathbf{C}$.

- We have an adjunction:

$$FG = (-) \otimes U \circlearrowright \mathbf{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathbf{C}||_u \quad (F \dashv G)$$

- The strength is then defined as:

$$T_v(A) \otimes U := FGT_v(A) \overset{\text{nat.}}{=} FT_uG(A) \overset{\text{unit}}{\to} FT_u{\color{magenta}GF}G(A) \overset{\text{nat.}}{=} {\color{blue}FG}T_vFG(A) \overset{\text{counit}}{\to} T_vFG(A) =: T_v(A \otimes U)$$

Back to Motivational Problems

Example: Concurrency

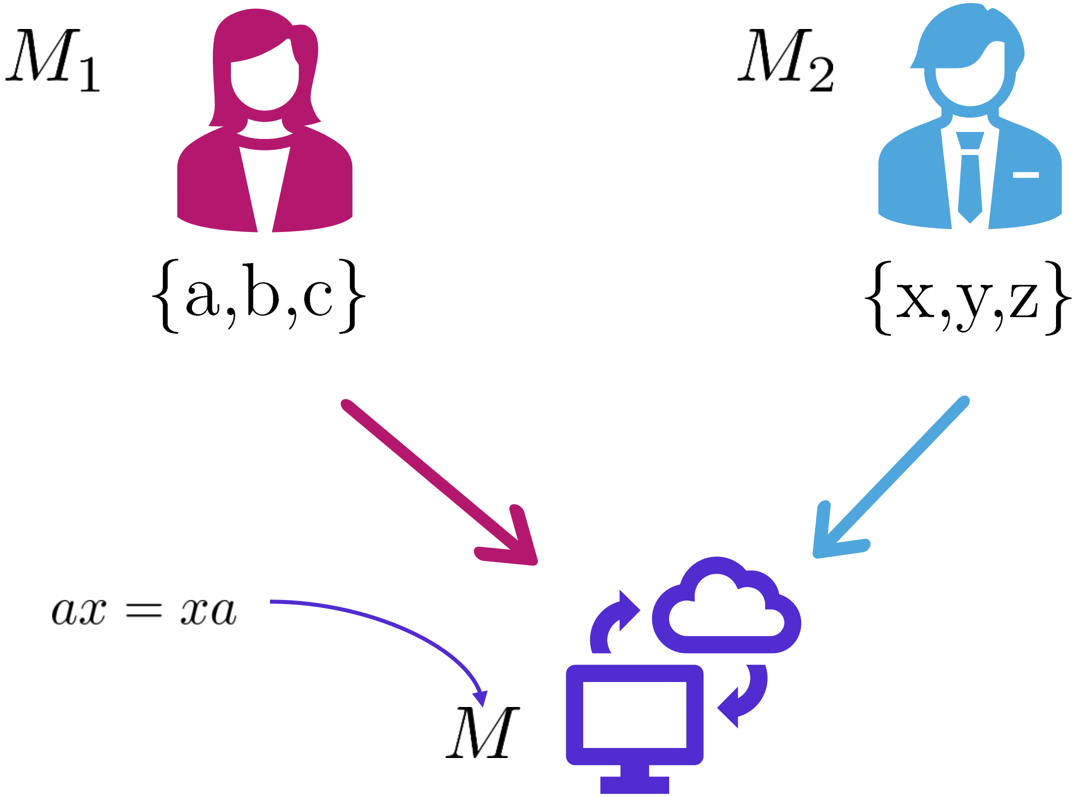
Monad (on **Set**)

$A \longmapsto M \times A$

(Action monad / Writer monad)

Kleisli maps

$A \longrightarrow M \times B$

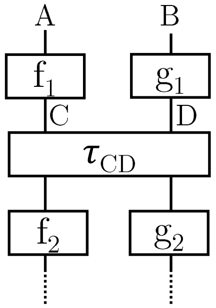


Start: Categories $\mathbf{C}_1$ and $\mathbf{C}_2$ with monoids M_1 and M_2.

Want: Category $\mathbf{C}$ with monoid M that "restricts" to $\mathbf{C}_1$ and $\mathbf{C}_2$ with monoids M_1 and M_2.

Building C:

- Idea:
  - Ob: pairs (A, B) of $A \in \mathbf{C}_1$ and $B \in \mathbf{C}_2$.
  - Mor:



- Central idempotents:

$$u_1 = (I, 0),\ u_2 = (0, I)$$

- In essence, we get a monoid M of the form

$$M = (M_1, M_2)$$

- Then we obtain the restrictions we wanted:
  - Restricting $\mathbf{C}$ to u_i yields $\mathbf{C}_i$, while M restricts to M_i.

Example: Localising the global state monad

- State monad on **Set**:

$$T(-) = S \multimap (- \times S)$$

Central idempotents: 0, 1

➢ This is trivially localisable!

What about something like this?

Central idempotents: 2^n

- State monad on $\mathbf{Set}^n$:

$$T(A_1, \ldots, A_n) = (S_1, \ldots, S_n) \multimap ((A_1, \ldots, A_n) \times (S_1, \ldots, S_n))$$

- Strength is curry of

$$T(A_1, \ldots, A_n)\times(U_1, \ldots, U_n)\times(S_1, \ldots, S_n) \longrightarrow (S_1, \ldots, S_n)\times(A_1, \ldots, A_n)\times(U_1, \ldots, U_n)$$

➢ Similar question on **Hilb** and $\mathbf{Hilb}^n$

Other things to mention

- Algebras

- Commutativity

$$
\begin{array}{ccccc}
T(A) \otimes U \otimes V & \xrightarrow{\mathrm{st}_{A,U} \otimes V} & T(A \otimes U) \otimes V & \xrightarrow{\mathrm{st}_{A \otimes U, V}} & T(A \otimes U \otimes V) \\
{\scriptstyle T(A) \otimes \sigma_{U,V}} \downarrow & & & & \uparrow {\scriptstyle T(A \otimes \sigma_{V,U})} \\
T(A) \otimes V \otimes U & \xrightarrow[\mathrm{st}_{A,V} \otimes U]{} & T(A \otimes V) \otimes U & \xrightarrow[\mathrm{st}_{A \otimes V, U}]{} & T(A \otimes V \otimes U)
\end{array}
$$

- Connections to other ideas of modularity on monads

Thank You!

Paper: https://arxiv.org/abs/2108.01756
(To appear at CSL 2022)