A Taste of Quantitative Logic

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What is quantitative logic?

In the words of Lawvere '73 paper on *generalised logics*:

> *"Logic signifies formal relationships which are general in character, we may more precisely identify logic with this scheme of interlocking adjoints and then observe that all of logic applies directly to structures valued in arbitrary closed categories V[, f]or example in quantitative logic (V = R)."*

Thus **quantitative logic is logic whose *judgments* are valued in R.**

Multiplicative Reals

Multiplicative structure of the positive reals

dist

polarity		additive			multiplicative
duality $a^* := 1/a$	**positive**	**false** $:= 0$ $\boxed{a \vee b}$	$\xleftarrow{p\to\infty}$ $a + b$	$\mathbf{0} := 0$ $a \oplus^p b$ $= (a^p + b^p)^{1/p}$	$\mathbf{1} := 1$ $a \otimes b$
	negative	**true** $:= \infty$ $a \wedge b$	$\xleftarrow{p\to\infty}$ $a +^* b = \frac{1}{\frac{1}{a} + \frac{1}{b}}$	$\top := \infty$ $a \oplus^{-p} b$ $= (a^{-p} + b^{-p})^{-1/p}$	$\bot := 1$ $a \otimes^* b$

$$\frac{\text{lim dist}}{a \otimes (b \otimes^* c) \leq (a \otimes b) \otimes^* c}$$

$a \otimes b$	0	$a \in (0,\infty)$	∞
0	0	0	0
$b \in (0,\infty)$	0	ab	∞
∞	0	∞	∞

$a \otimes^* b$	0	$a \in (0,\infty)$	∞
0	0	0	∞
$b \in (0,\infty)$	0	ab	∞
∞	∞	∞	∞

The *doxastic interpretation* of real-valued alethics

We think of real numbers as **ratios of belief**. Then

I. To believe a fact **1** means ‘I am indifferent to it’,
II. To believe a fact **0**/**∞** means ‘I believe it is impossible/certain’
III. To believe a fact **a** means ‘I am **a** times as confident in it than baseline’,
IV. To believe a fact **a*** means ‘I am **a** times as dubious in it than baseline’
V. To believe a fact **a + b** means ‘I have reasons to believe it **a** and different reasons to believe it **b**’, $0 \xrightarrow{p} \infty$
VI. To believe a fact **a ⊗ b** means ‘it is a combination of a fact I believe **a** and another fact I believe **b**’.

Note: nothing to say about *why* I believe some atomic fact to some degree.

$$
\begin{tikzcd}[column sep=huge]
 & \textbf{multiplicative} \arrow[r, leftrightarrow, "\textbf{Napierian duality}"] & \textbf{additive} \\
\textbf{negative} \arrow[dd, leftrightarrow, "\textbf{de Morgan duality}"'] & \boxed{[0,1]_{\otimes}} \arrow[d, hook] \arrow[r, shift right, "-\log"'] \arrow[dd, dotted, bend right=60, leftrightarrow, "1/-"'] & {[-\infty,0]^{\mathrm{op}}_{\oplus}} \arrow[l, shift right, "1/\exp"'] \arrow[l, phantom, "\sim"] \arrow[d, hook] \arrow[dd, dotted, bend left=60, leftrightarrow, "-(-)"] \\
 & \boxed{[0,\infty]_{\otimes}} \arrow[r, shift right, "-\log"'] & {[-\infty,\infty]^{\mathrm{op}}_{\oplus}} \arrow[l, shift right, "1/\exp"'] \arrow[l, phantom, "\sim"] \\
\textbf{positive} & {[1,\infty]_{\otimes}} \arrow[u, hook] \arrow[r, shift right, "-\log"'] & {[0,\infty]^{\mathrm{op}}_{\oplus}} \arrow[u, hook] \arrow[l, shift right, "1/\exp"'] \arrow[l, phantom, "\sim"]
\end{tikzcd}
$$

probability theory

$a + b = \log(e^a + e^b)$

Lawvere

Bacci
Mardare
Pan... Plotkin

Propositional quantitative logic

Quantitative calculi

$$\frac{\Gamma \vdash \Delta \quad \cdots \quad \Gamma' \vdash \Delta'}{\Gamma'' \vdash \Delta''}$$

Definition III.1. Structures of sequents are defined inductively:

$$\begin{aligned} \mathcal{H} \quad ::= \quad & \Gamma \vdash \Delta \mid \mathbf{1} \mid \mathbf{0} \mid \infty \\ \mid \quad & \mathcal{H} \otimes \mathcal{H} \mid \mathcal{H} \oplus^{p} \mathcal{H} \mid \mathcal{H} \oplus^{-p} \mathcal{H} \end{aligned} \tag{16}$$

where p ranges in $(0, \infty]$. A structure of sequents involving no sequent is called **closed**, while a structure of sequents consisting of a single sequent is called **unary**.

A **rule** has thus the form $\frac{\mathcal{H}}{\mathcal{H}'}$. It is an **axiom** if the top structure of sequents is closed. A set of rules $\mathscr{R}$ is a **calculus**. A **derivation** in the calculus $\mathscr{R}$ is defined inductively as follows.

1) Every rule in $\mathscr{R}$ is a derivation,
2) for each structure of sequents the **identity rule** $\frac{\mathcal{H}}{\mathcal{H}}\,\mathcal{H}$ is a derivation, and
3) if $\frac{\mathcal{H}_1 \bullet \mathcal{H}_2}{\mathcal{K}}\,R$ is a rule in $\mathscr{R}$ where $\bullet \in \{\otimes, \oplus^{p}, \oplus^{-p}\}$, and if $\begin{array}{c}\vdots\, D_1 \\ \mathcal{H}_1\end{array}$ and $\begin{array}{c}\vdots\, D_2 \\ \mathcal{H}_2\end{array}$ are derivations, then $\begin{array}{c}\vdots\, D_1 \bullet D_2 \\ \dfrac{\mathcal{H}_1 \quad \bullet \quad \mathcal{H}_2}{\mathcal{K}}\,R\end{array}$ is a derivation.

Quantitative calculi

$[0, \infty]$

$$\begin{array}{c} (1 \oplus^{P} 1) \otimes \infty \\ \hline \vdots \\ \hline \Gamma \vdash \Delta \end{array}$$

$$\begin{array}{c} 0 \\ \hline \vdots \\ \hline \Gamma \vdash \Delta \end{array}$$

A **closed** derivation is a derivation where the top structure of sequents is closed. A **proof** is a closed derivation where the bottom structure of sequents is unary. The **validity** $|P|$ of a closed derivation P is the alethic value of the top structure of sequents. The **provability** $|\Gamma \vdash \Delta|_{\mathscr{R}}$ of a sequent $\Gamma \vdash \Delta$ in the calculus $\mathscr{R}$ is the supremum of the validity of its closed derivations:

$$|\Gamma \vdash \Delta| = \bigvee \left\{ |P| \;\middle|\; \begin{array}{c} \vdots \, P \\ \Gamma \vdash \Delta \end{array} \right\} \tag{17}$$

QLL

Quant Linear Logic

(a) *Structural Rules:*

$$\frac{\Gamma, \Xi \vdash \Delta}{\Xi, \Gamma \vdash \Delta}\ \mathrm{EX_L} \qquad \frac{\mathbf{1}}{\Gamma \vdash \Gamma}\ \mathrm{AX} \qquad \frac{\Gamma \vdash \Xi, \Delta}{\Gamma \vdash \Delta, \Xi}\ \mathrm{EX_R}$$

$$\frac{\Gamma' \vdash A, \Delta' \quad \otimes \quad \Gamma, A \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathrm{CUT}$$

QLL

(b) *Multiplicative Fragment:*

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes_{\mathrm{L}} \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \otimes^* B}\ \otimes^*_{\mathrm{R}}$$

$$\frac{\Gamma, A \vdash \Delta \quad \otimes \quad B, \Gamma' \vdash \Delta'}{\Gamma, A \otimes^* B, \Gamma' \vdash \Delta, \Delta'}\ \otimes^*_{\mathrm{L}} \qquad \frac{\Gamma \vdash \Delta, A \quad \otimes \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash \Delta, A \otimes B, \Delta'}\ \otimes_{\mathrm{R}}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta}\ \mathbf{1}_{\mathrm{L}} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^* \vdash \Delta}\ {}^*_{\mathrm{L}} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^*, \Delta}\ {}^*_{\mathrm{R}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta}\ \mathbf{1}_{\mathrm{R}}$$

$$\frac{\Gamma \vdash \Delta \quad \otimes \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathrm{MIX}$$

QLL

$$\frac{|\,1€ \vdash \text{Beer}\,| = 1/5 \;\oplus^{A}\; |\,1€ \vdash \text{Beer}\,| = 1/4}{|\,1€ \vdash \text{Pizza} \otimes^{*} \text{Beer}\,| = 1/9}$$

(c) *Additive Fragment:*

$$\frac{\Gamma, A \vdash \Delta \quad \oplus^{-p} \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee^p B \vdash \Delta} \vee^p{}_{\mathrm{L}} \qquad \frac{\Gamma \vdash \Delta, A \quad \oplus^{-p} \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge^p B} \wedge^p_{\mathrm{R}}$$

$$\frac{\Gamma, A \vdash \Delta \quad \oplus^{p} \quad \Gamma, B \vdash \Delta}{\Gamma, A \wedge^p B \vdash \Delta} \wedge^p_{\mathrm{L}} \qquad \frac{\Gamma \vdash A, \Delta \quad \oplus^{p} \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee^p B, \Delta} \vee^p_{\mathrm{R}}$$

$$\frac{\infty}{\Gamma, \bot \vdash \Delta} \bot_{\mathrm{L}} \qquad \frac{\mathbf{0}}{\Gamma \vdash \Delta} \mathrm{EFQ} \qquad \frac{\infty}{\Gamma \vdash \top, \Delta} \top_{\mathrm{R}}$$

$!_p A$

$\Gamma \vdash \Delta \; \underline{\text{IMALL}} \iff |\Gamma \vdash \Delta|_{\text{QLL}} \geq 1$

TH

$\mathbb{R} \times \mathbb{R} \xrightarrow{\bullet} \mathbb{R}$

(1) $\bullet$ idem

(2) $\bullet$ ass

(3) "smooth monotonicity"

$\partial_i \bullet > 0$

$\oplus^{-p} / \oplus^{p}$

$$\dfrac{\dfrac{1}{A \vdash A} \quad \oplus^{-p} \quad \dfrac{1}{A \vdash A}}{A \vdash A \wedge^p A} \wedge^p_R$$

$$\dfrac{\begin{array}{c} \vdots\, p_1 \\ \Gamma \vdash \Delta \end{array} \quad \oplus \quad \begin{array}{c} \vdots\, p_2 \\ \Gamma \vdash \Delta \end{array}}{\Gamma \vdash \Delta}$$

$\frac{1}{2} \otimes (\Gamma \vdash \Delta \oplus \Gamma \vdash \Delta)$

$\rightsquigarrow 2^{-1/p}$ "degree of idempotency of $\wedge^p$"

Algebraic semantics

Definition 2.26 (Softale). A *softale* is a $[0,\infty]_\otimes$-enriched preorder $(\mathcal{S},\sqsubseteq)$ equipped with:

- a symmetric monoidal structure $(u,\bullet)$ on the underlying preorder, satisfying

$$(a \sqsubseteq b)\otimes(c \sqsubseteq d) \;\leq\; (a\bullet c)\sqsubseteq(b\bullet d),$$

$$a\bullet u \cong a, \qquad u\bullet a\cong a, \qquad (a\bullet b)\bullet c\cong a\bullet(b\bullet c), \qquad a\bullet b\cong b\bullet a;$$

- a duality $(-)^\perp$ on objects, involutive and satisfying

$$\mathcal{S}(a,b)=\mathcal{S}(b^\perp,a^\perp), \qquad \mathcal{S}(a\bullet b,c^\perp)\cong\mathcal{S}\big(a,(b\bullet c)^\perp\big);$$

- for every $p\in\mathcal{H}$, all p-meets and p-joins, i.e. objects $a\wedge^p b$, $a\vee^p b$ such that

$$(-\sqsubseteq a\wedge^p b)\cong(-\sqsubseteq a)\oplus^{-p}(-\sqsubseteq b), \qquad (a\vee^p b\sqsubseteq -)\cong(a\sqsubseteq -)\oplus^{-p}(b\sqsubseteq -);$$

  including a terminal object $\top$ and initial object $\perp$:

$$(a\sqsubseteq\top)\cong\infty, \qquad (\perp\sqsubseteq a)\cong 0.$$

  satisfying the distributivity axioms

$$\perp\cong a\bullet\perp, \qquad (a\bullet b)\vee^p(a\bullet c)\;\cong\; a\bullet(b\vee^p c).$$

Construction IV.19. The **syntactic softale** $\mathcal{F}$ is defined as follows:

1) its elements are the formulae of Definition III.2,
2) its order relation is defined as

$$\varphi\Rightarrow\psi := |\varphi\vdash\psi|\,, \tag{43}$$

   where the right hand side denotes the provability of a sequent (Equation (17));
3) its monoidal, de Morgan, and lattice structures are given by the connectives of the logic, i.e. respectively by $\otimes$, $(-)^*$, and $\wedge^p/\vee^p$.