TOPOS Institute - Oxford Seminar

The Path-Complete Formalism for Switched Systems: Stability and Beyond

UNIVERSITY OF OXFORD

DEPARTMENT OF COMPUTER SCIENCE

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April, 2025

captured by

real-world phenomena

mathematical models

(dynamical systems, MDPs, etc.)

captured by

provide guarantees

real-world phenomena

mathematical models

(dynamical systems, MDPs, etc.)

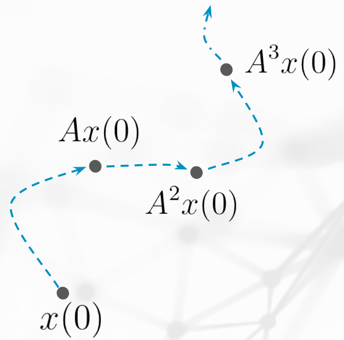
certificates

(stability, safety, etc.)

A discrete-time linear dynamical system is defined by

$$x(k+1) = Ax(k)$$

where $A \in \mathbb{R}^{n \times n}$ is called the dynamics of the system
$x(0) \in \mathbb{R}^n$ is the initial condition



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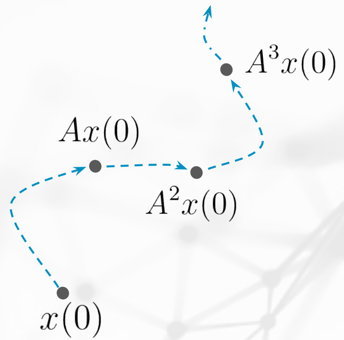

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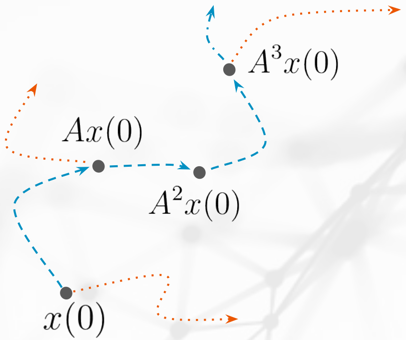

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A discrete-time linear **switched** dynamical system is defined by

$$x(k+1) = A_{\sigma(k)}x(k)$$

where $\Sigma := \{A_1, \ldots, A_M\} \subset \mathbb{R}^{n\times n}$ are the dynamics of the system
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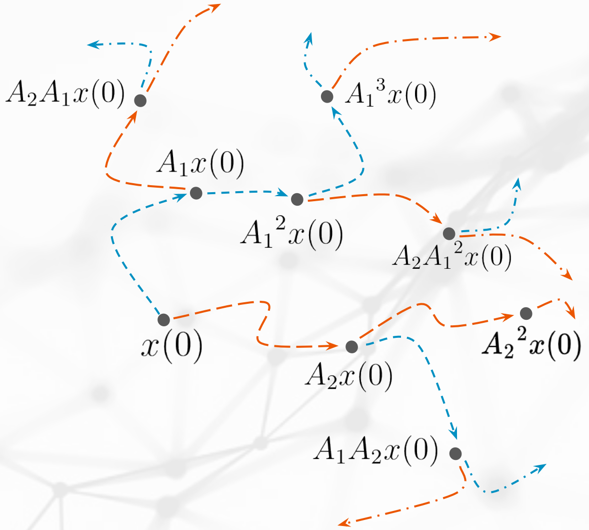

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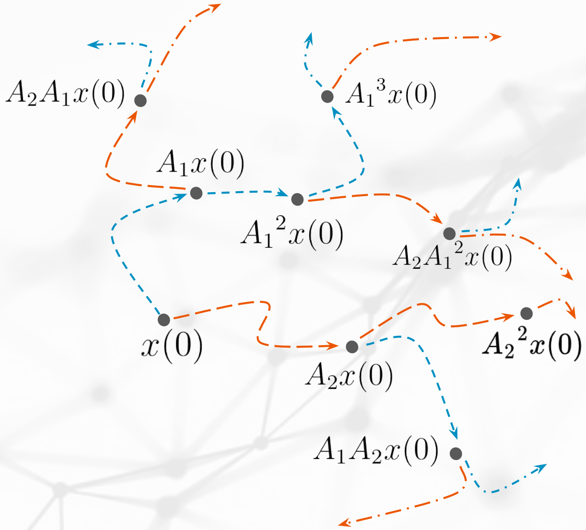
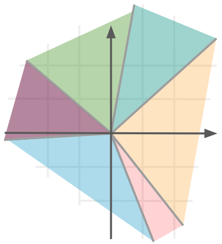




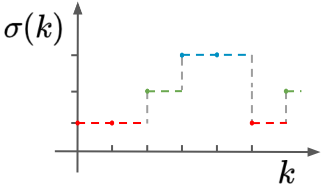
state-dependent switching signal

$$\sigma : \mathbb{R}^n \to \{1, \ldots, M\}$$
$$\forall k \in \mathbb{N},\ x(k) \mapsto \sigma(x(k))$$

VS

time-dependent switching signal

$$\sigma : \mathbb{N} \to \{1, \ldots, M\}$$
$$\forall k \in \mathbb{N},\ k \mapsto \sigma(k)$$



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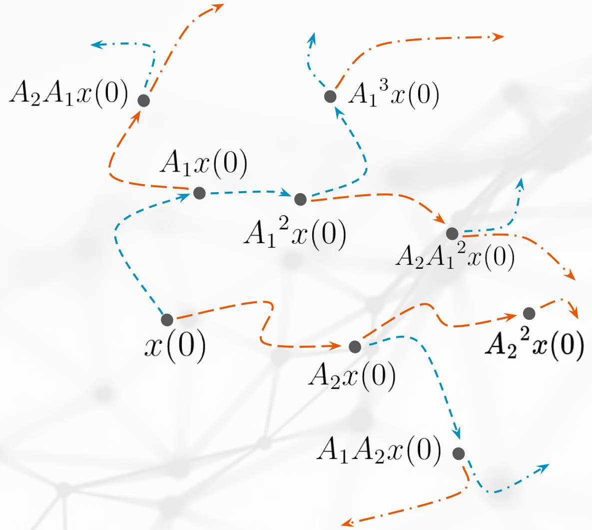
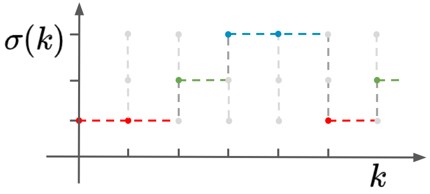


arbitrary switching signal

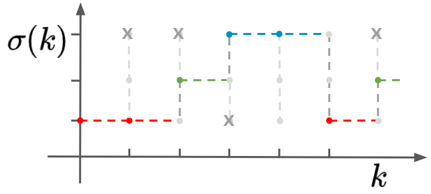


$\sigma \in \{1, \ldots, M\}^+$

VS

constrained switching signal

$\sigma \in \mathcal{L} \subset \{1, \ldots, M\}^+$



Part I - *Motivations and background*

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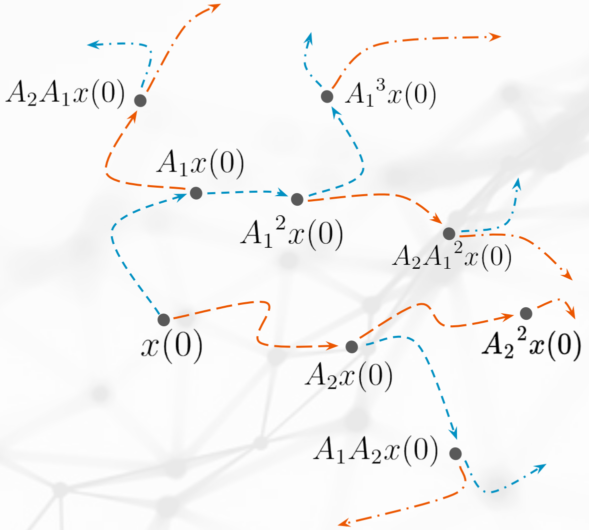

Definition. A discrete-time linear switched system $x(k+1) = A_{\sigma(k)}x(k)$ is asymptotically **stable under arbitrary switching** if for any initial condition $x(0) \in \mathbb{R}^n$ and any switching signal $\sigma : \mathbb{N} \to \{1, \dots, M\}$:

$$\lim_{k\to\infty} \|x_\sigma(k)\| = \lim_{k\to\infty} \left\| \cdots A_{\sigma(2)}A_{\sigma(1)}x(0) \right\| = 0$$

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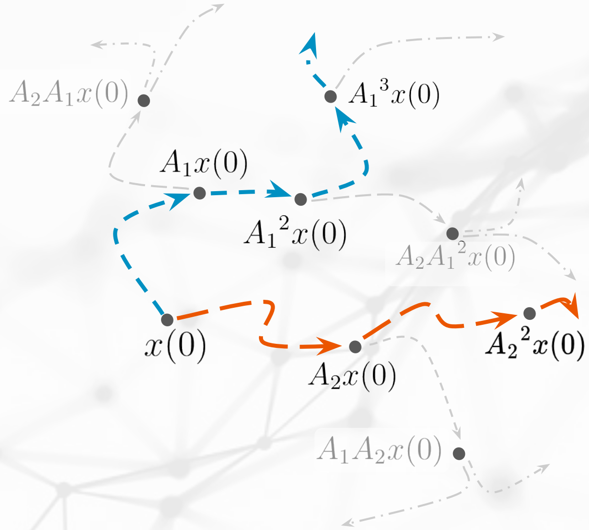

Proposition. For any finite set of matrices $\Sigma := \{A_1, \ldots, A_M\} \subset \mathbb{R}^{n\times n}$,

$x(k+1) = A_{\sigma(k)}x(k)$ is asympt. stable $\Rightarrow$ $\forall i = 1, \ldots, M,\ x(k+1) = A_i x(k)$ is asympt. stable

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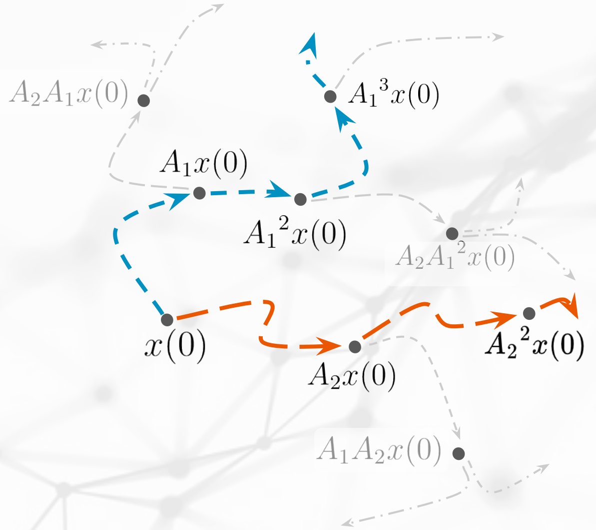

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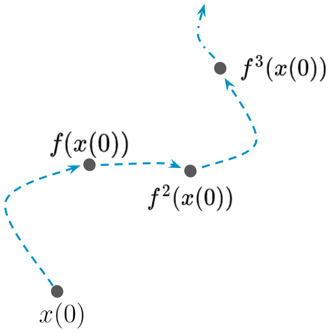
How to prove stability? Using Lyapunov theory!

$$x(k+1) = f(x(k))$$

Lyapunov theory : $\exists V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which decreases along any trajectory of the system, i.e.

$$\forall x(0) \in \mathbb{R}^n,\ V(x(k+1)) \leq V(x(k))\ \forall k \in \mathbb{N}$$

the initial state



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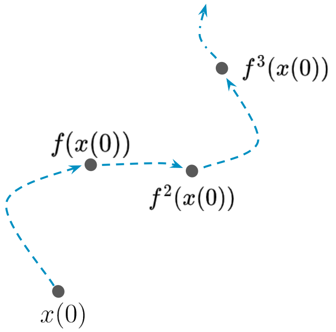

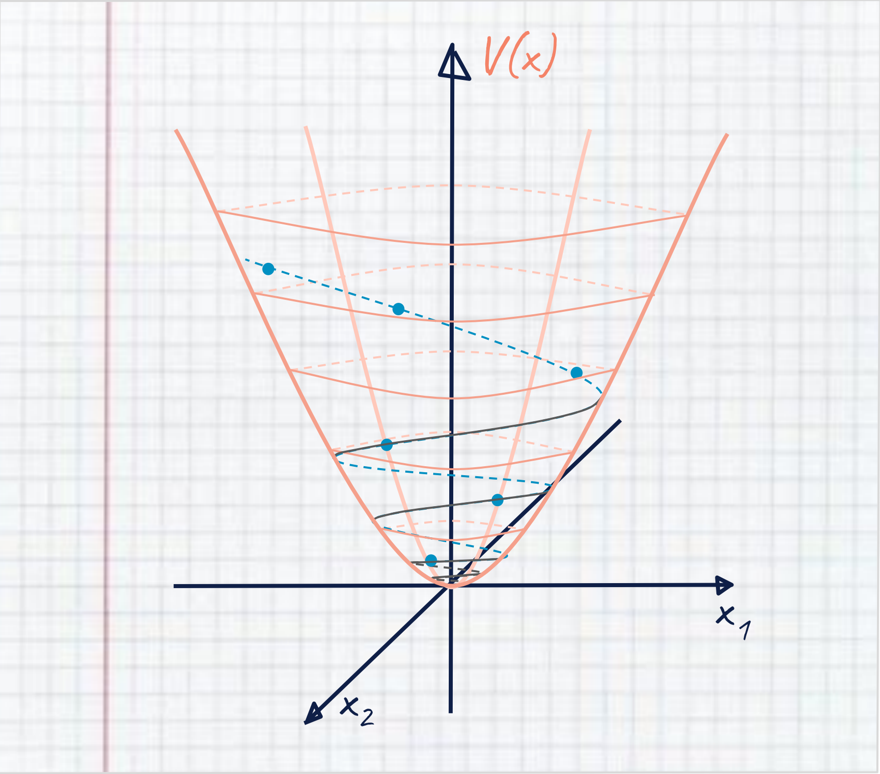

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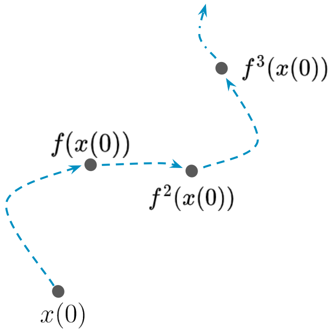
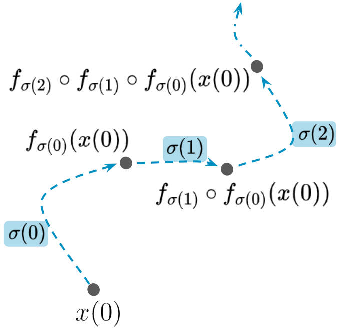


$$x(k+1) = f_{\sigma(k)}(x(k))$$

Generalization: $\exists V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which decreases along any trajectory of the system, i.e.

$$\forall x(0) \in \mathbb{R}^n, \forall \sigma : \mathbb{N} \to \langle M \rangle,$$
$$V(x_\sigma(k+1)) \leq V(x_\sigma(k))\ \forall k \in \mathbb{N}$$

the initial state **and a switching signal**



Definition. A **Common Lyapunov Function** (CLF), i.e. a continuous positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which satisfies the following **Lyapunov inequalities**:

$$\forall \sigma \in \{1, \ldots, M\}, \forall x \in \mathbb{R}^n : V(f_\sigma(x)) \leq V(x)$$

necessary and sufficient condition for stability

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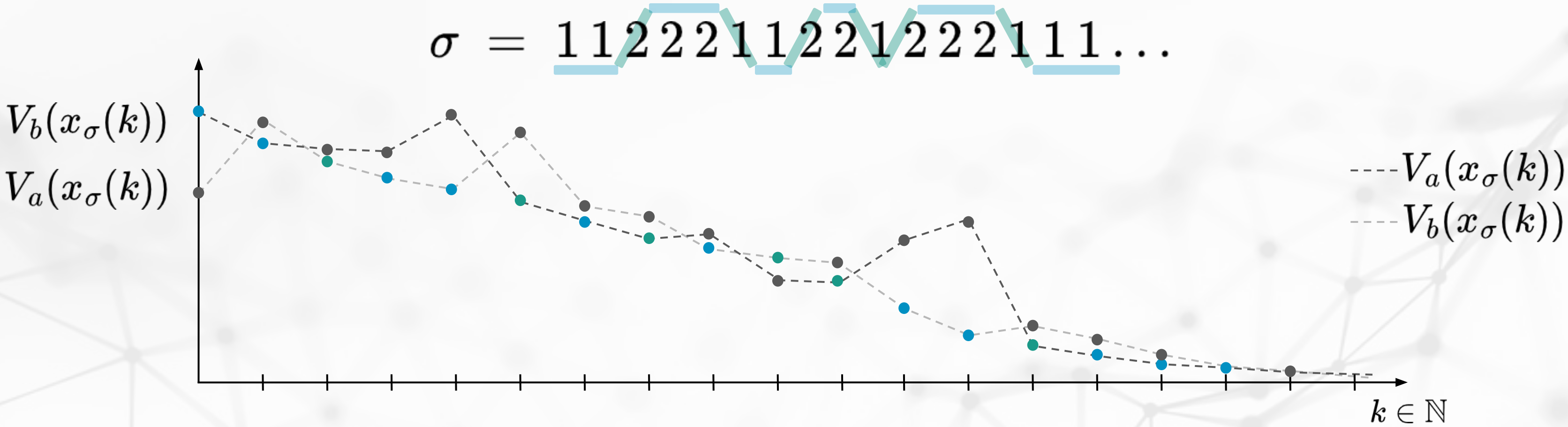
But numerically hard to find... which leads to ***Multiple Lyapunov Functions*** (MLF)

Some references [*] have introduced **Multiple Lyapunov functions**. For a switched system with $M = 2$ modes, the following inequalities imply the stability:

$$V_a\left(f_1(x)\right) \leq V_a(x) \qquad V_b\left(f_1(x)\right) \leq V_a(x)$$

$$V_b\left(f_2(x)\right) \leq V_b(x) \qquad V_a\left(f_2(x)\right) \leq V_b(x)$$

[*] J. Daafouz and J. Bernussou, 2001.



Both act as a Lyapunov function when the **switching signal is constant**

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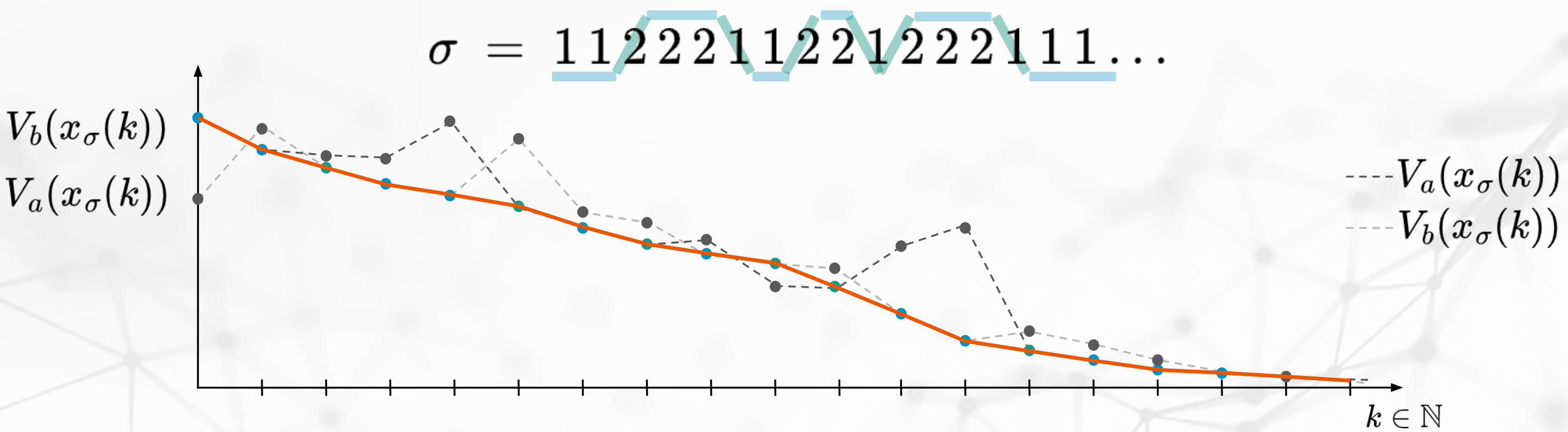

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Previous sets of **Lyapunov functions** can be encoded by **directed and labeled graphs:**

Common Lyapunov function

a ***single*** *Lyapunov function that decreases along every mode of the switching system*

$$\forall i \in \{1, 2\},\ V(f_i(x)) \leq V(x)$$

Multiple Lyapunov function

a ***set*** *of Lyapunov functions whose decrease properties cover every switching sequence*

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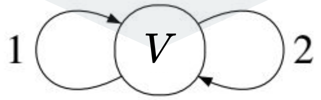

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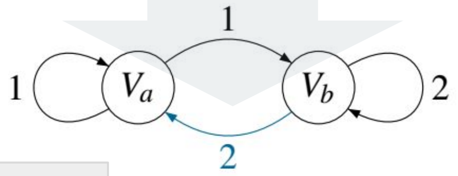

a **node** $\longleftrightarrow$ a Lyapunov **function**
an **edge** $\longleftrightarrow$ a Lyapunov **inequality**

A **path** on a directed and labeled graph ⟷ a **switching sequence** ⟷ a successive **combination of dynamics**

$$p = e_1 e_2 e_3 e_4 \dots \qquad \sigma = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \dots \qquad f_\sigma = f_{\sigma_1} f_{\sigma_2} f_{\sigma_3} f_{\sigma_4} \dots$$

Definition. A graph is **path-complete** if any finite switching sequence can be produced by the labels of a sequence of edges in the graph. (*i.e. the language of the graph is the Kleene closure of the alphabet*)

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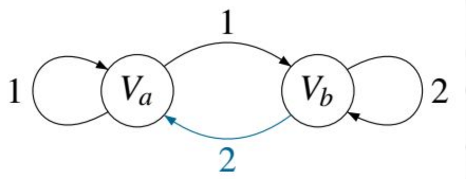
Example:

Path-complete graph

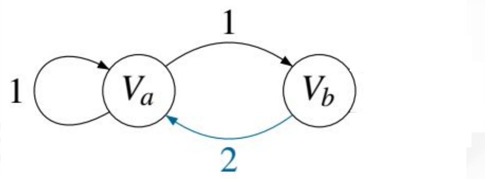
Any sequence can be generated with the labels



$\sigma = 121211221...$

NOT Path-complete graph

This graph cannot create a sequence with consecutive "2"

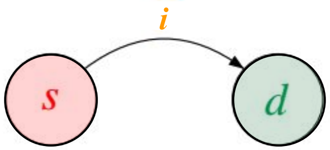


Given a linear switched system $\Sigma := \{A_1, \ldots, A_M\} \subset \mathbb{R}^{n \times n}$:

Definition. A **Path-Complete Lyapunov Function** (PCLF) is a pair $(\mathcal{G}, V)$ where

- the **graph** $\mathcal{G} = (S, E)$ is **path-complete**,
- a candidate Lyapunov function $\{V_s : s \in S\} \in \mathcal{V}^S$ in a **template** $\mathcal{V}$,

such that the following Lyapunov inequalities are satisfied



$$\forall (s, d, i) \in E, \forall x \in \mathbb{R}^n : \; V_d(A_i x) \leq V_s(x)$$

We denote it by $V \in PCLF(\mathcal{G}, \Sigma)$.

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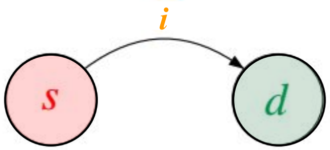

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Theorem. A PCLF is a sufficient condition for stability.

a **combinatorial** component
the ***graph***

an **algebraic** component
the ***template***

the **structure** of the Lyapunov inequalities is encoded by a *path-complete* **graph**

Each graph is a sufficient condition for

$$\forall x(0) \in \mathbb{R}^n, \forall \sigma : \mathbb{N} \to \langle M \rangle, \quad V(x_\sigma(k+1)) \leq V(x_\sigma(k)) \; \forall k \in \mathbb{N}$$

- encodes in particular the classical **Common** Lyapunov Function (CLF)
- generalizes the concept of **Multiple** Lyapunov Functions (MLFs)

the **nature** of the Lyapunov functions is encoded by a **template**

- **quadratic** template

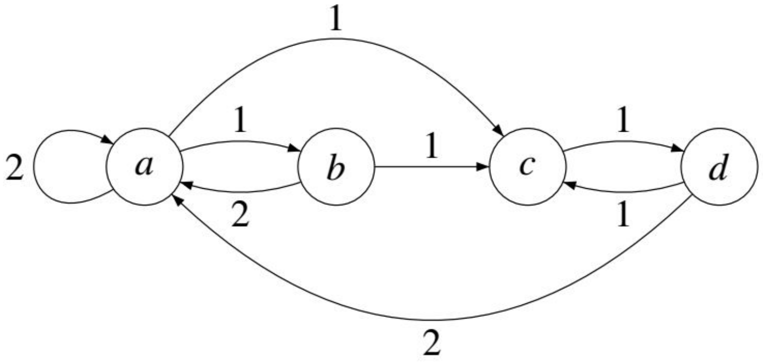
$$\mathcal{Q} := \{V(x) := x^\top P x \mid P \succ 0\}$$

- template of primal **linear copositive norms**

$$\mathcal{L} := \{V(x) := v^\top x \mid v \geq_c 0\}$$

The graph structure determines how the Lyapunov pieces must be compose:

$$\sigma = 1212112211121\ldots$$



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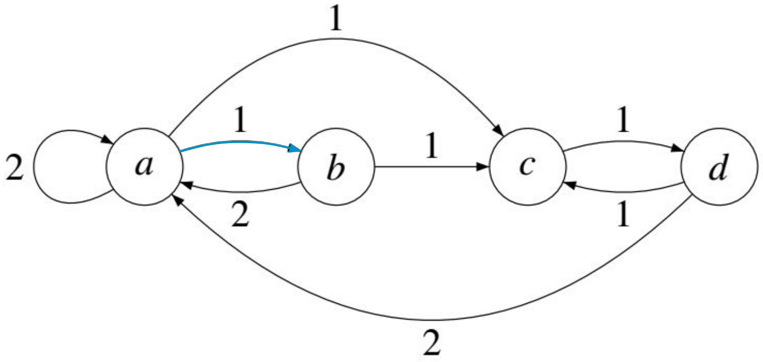

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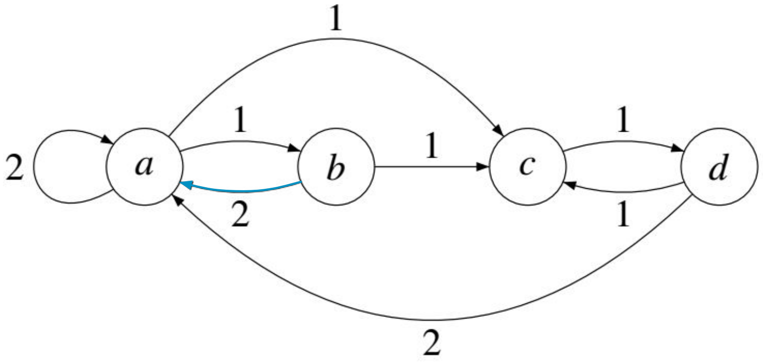

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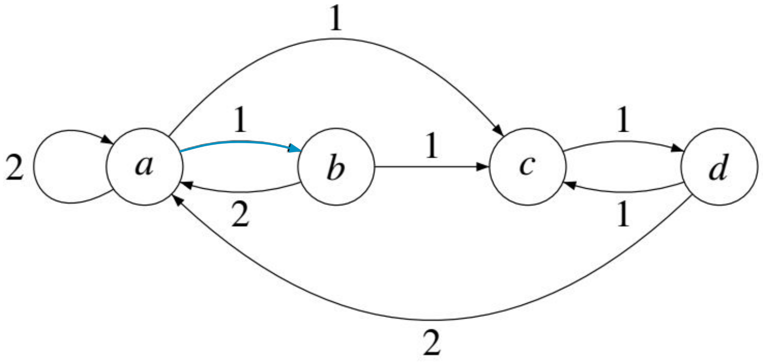

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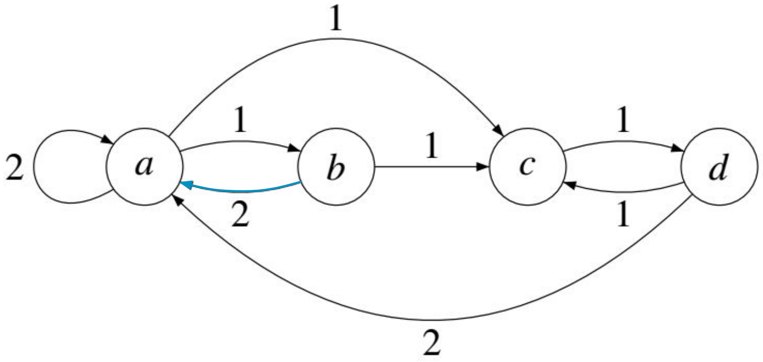

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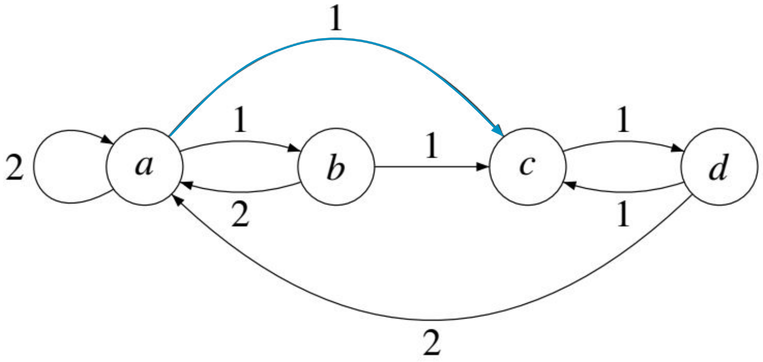

$$V_a(x_0) \geq V_b(x_1) \geq V_a(x_2) \geq V_b(x_3) \geq V_a(x_4) \geq V_c(x_5)$$

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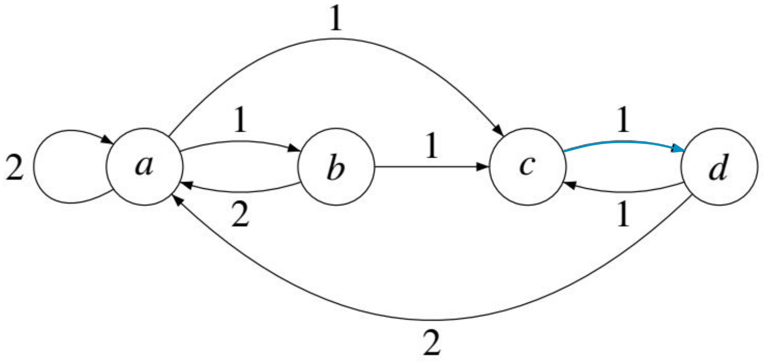
$$\sigma = 121211221 1121\ldots$$



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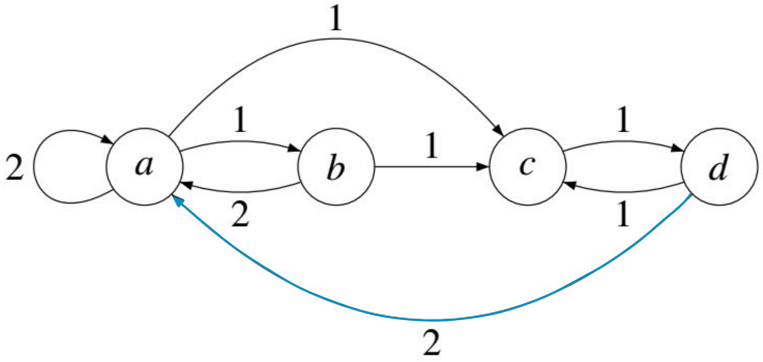

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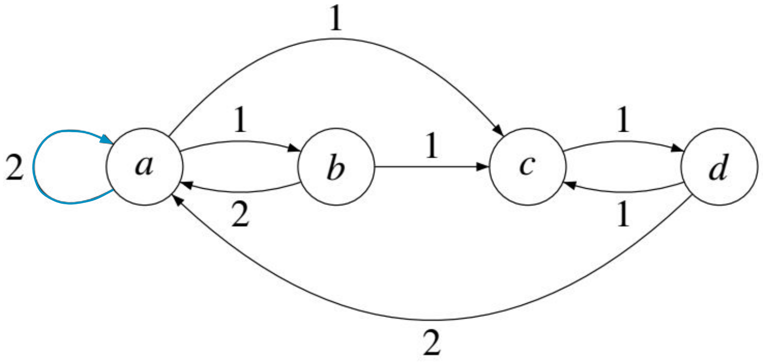

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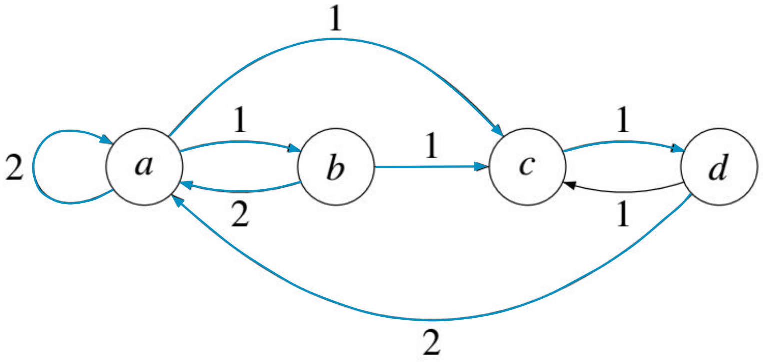

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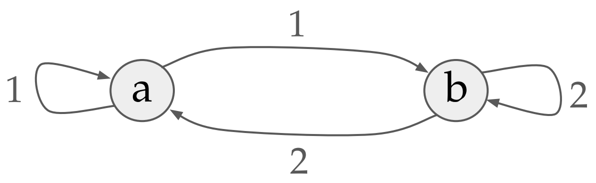

$$V_a(x_1, x_2) = 5{x_1}^2 + {x_2}^2$$
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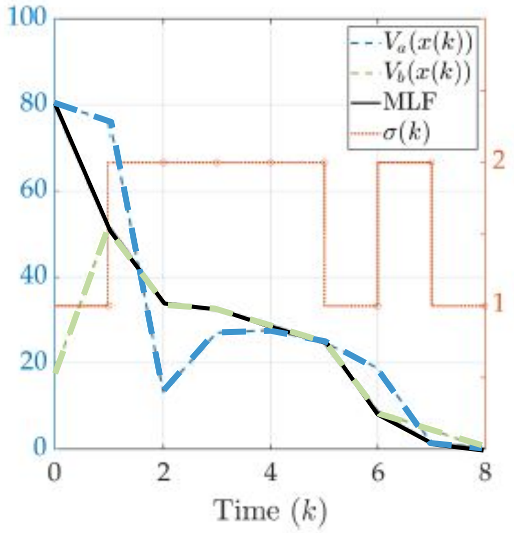

Evolution with time k of the switching signal σ (on the right axis), Lyapunov pieces Va and Vb, and the multiple Lyapunov function (on the left axis) along the trajectory starting at $x(0) = [4, -1/2]^\top$ and following the switching sequence $\sigma := 122221212$.

A few results in bulk:

A Path-complete Lyapunov function is a **sufficient and necessary** (for some templates and systems) **condition** for stability.

A.A. Ahmadi, R.M. Jungers, P.A. Parrilo and M. Roozbehani, SIAM, 2014

V. Debauche, M. Della Rossa and R.M. Jungers, NAHS, 2022

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Part II - *Path-complete Lyapunov functions*

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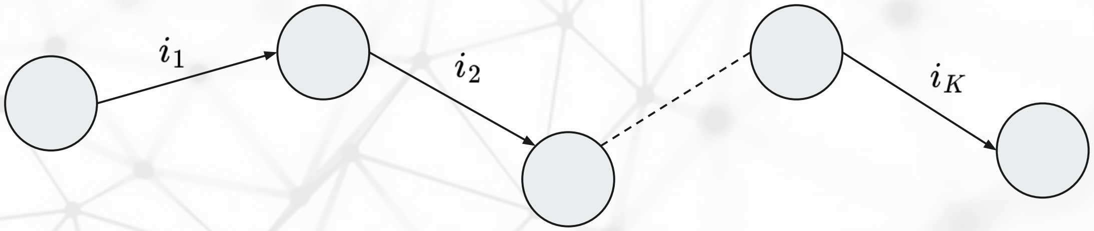
A set of **Lyapunov inequalities** is a sufficient condition for stability if and only if the **corresponding graph is path-complete**.

R. Jungers, A. A. Ahmadi, P. A. Parrilo and M. Roozbehani, 2017

$$w := i_1 i_2 \dots i_K \in \langle M \rangle^+$$



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M. Philippe, R. Essick, G. Dullerud and R. Jungers, 2015

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THEOREM 1 (INDUCED COMMON LYAPUNOV FUNCTION). *Consider Path-Complete Lyapunov function with graph* $\mathbf{G} = (S, E)$ *and pieces* $\mathcal{V} = (V_s)_{s \in S}$ *for the system* (1). *Let* $O^\star(\mathbf{G}) = (S^\star_O, E^\star_O)$ *be the complete and connected sub-graph of the observer* $O(\mathbf{G})$. *Then, the function*

$$V(x) = \min_{Q \in S^\star_O} \left(\max_{s \in Q} V_s(x) \right) \qquad (5)$$

is a Common Lyapunov function *for the system* (1).

REMARK 3. *We can establish a 'dual' version of the Theorem 1. In specific, given the graph* $\mathbf{G}$*, we reverse the direction of the edges obtaining a graph* $\mathbf{G}^\top$*, construct its observer* $O(\mathbf{G}^\top)$ *and reverse the direction of its edges again, obtaining a graph* $O(\mathbf{G}^\top)^\top$. *This graph is co-deterministic and contains a unique, strongly-connected, co-complete subgraph that induces a Lyapunov function of the form*

$$V(x) = \max_{S_1, \ldots, S_k \subseteq S} \left(\min_{s \in S_i} V_s(x) \right),$$

which is, in general, not *equal to the common Lyapunov function obtained through Theorem 1.*

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The path-complete formalism relies on the following key idea:

The path-completeness theory ensures that a property is guaranteed by **propagating it through the edges** of a graph. The expressivity of the graph characterizes the language (set of switching signals) for which this property holds.

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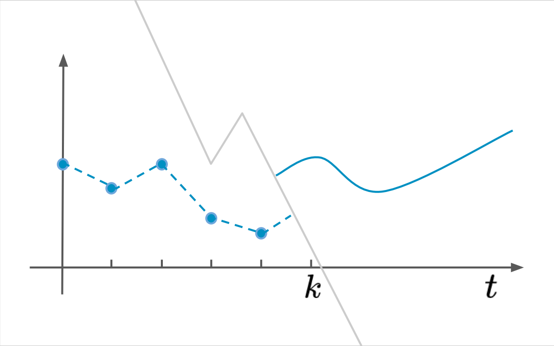
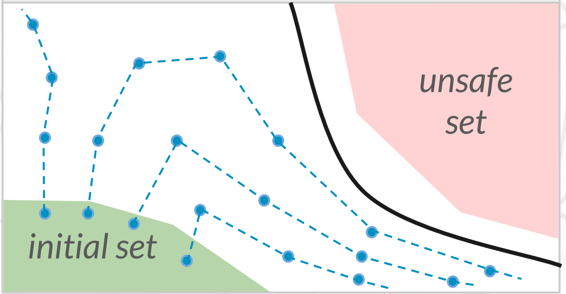
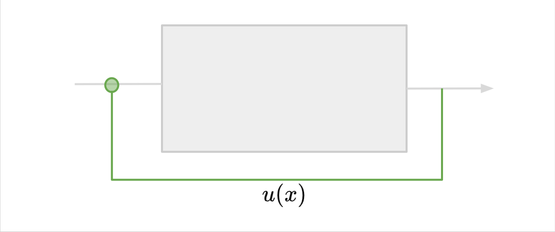
from discrete to **continuous** time



from stability to **safety** using path-complete **Barrier** functions



from stability to **stabilisation** using **control** path-complete **Lyapunov** functions



Continuous time: Given a **continuous**-time linear switched system is of the form

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A **Path-Complete Lyapunov Function** (PCBF) is a pair $(\mathcal{G}, V)$ where

- the **graph** $\mathcal{G} = (S, E)$ is **path-complete**,
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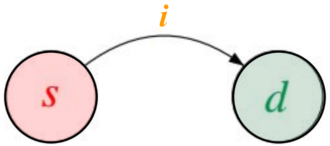
such that the following Lyapunov inequalities are satisfied

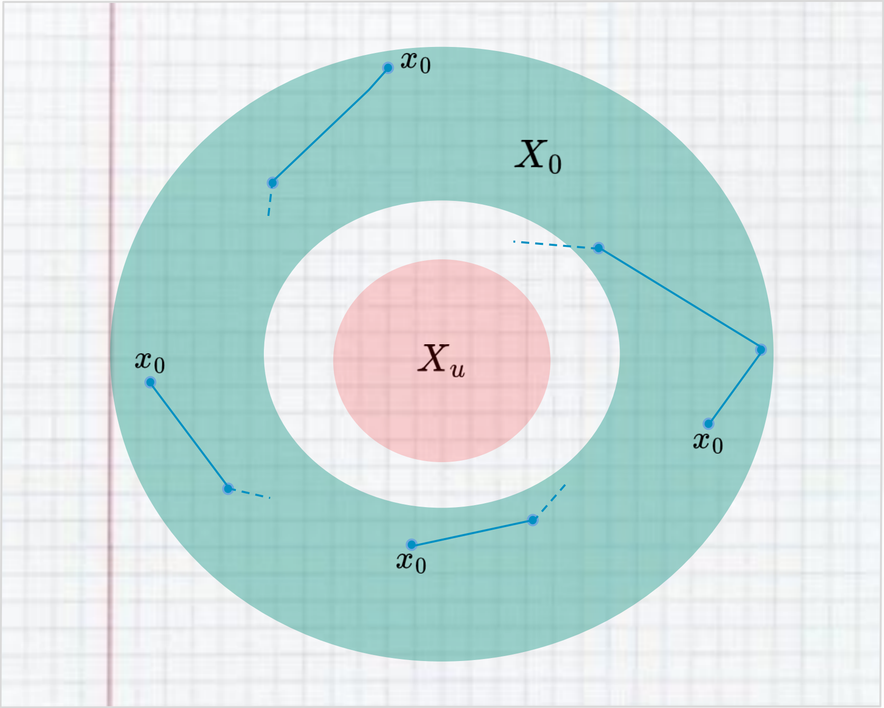


$$\forall x \in X, \forall (s,d,i) \in E, \begin{cases} \langle \nabla V_s(x), f_i(x) \rangle & \leq & \frac{1}{\tau}(V_s(x) - V_d(x)) \\ \langle \nabla V_d(x), f_i(x) \rangle & \leq & \frac{1}{\tau}(V_s(x) - V_d(x)) \end{cases}$$

This implies global stability on $\mathcal{S}_{fix}(\tau) := \left\{ \sigma \mid \frac{t_i^\sigma - t_{i-1}^\sigma}{\tau} \in \mathbb{N}, \forall t_i^\sigma > 0 \right\}$

M.Anand, R. Jungers, M. Zamani & F. Allgöwer. Path-Complete Barrier Functions for Safety of Switched Linear Systems, CDC 2024.

Safety analysis: Given an initial set $X_0 \subseteq X$ and an unsafe set $X_u \subseteq X$, the system is *safe* if any trajectory starting in X_0 never enter the unsafe set



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$$x(k+1) = f(x(k))$$

Barrier function theory : $\exists B : \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall x \in X_0, \quad B(x) \leq 0$$

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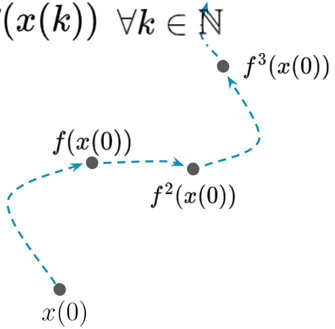
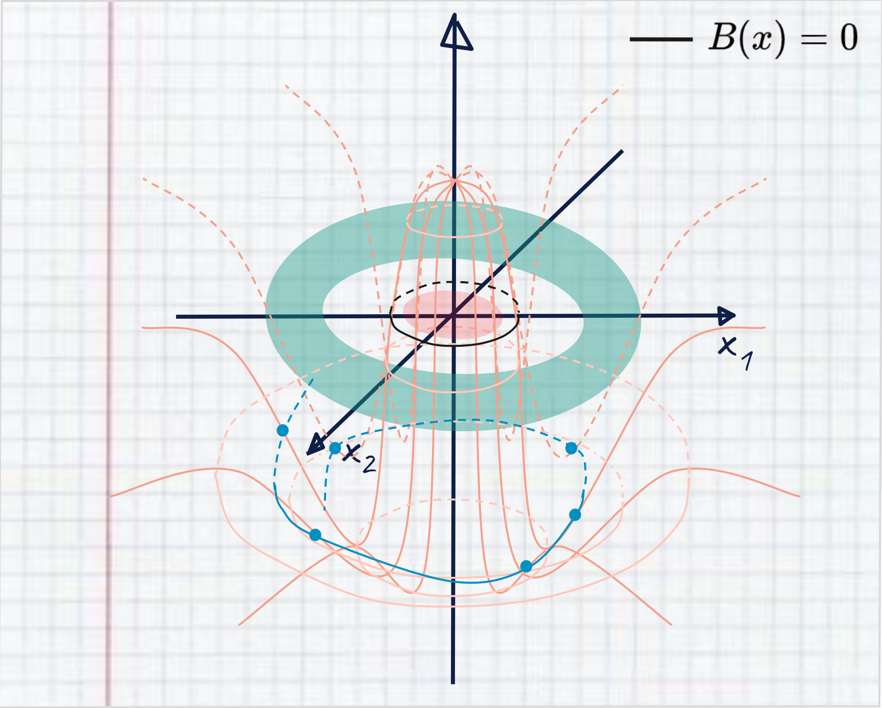
the initial state





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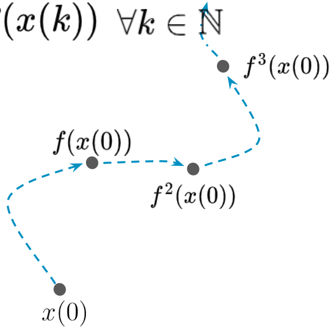

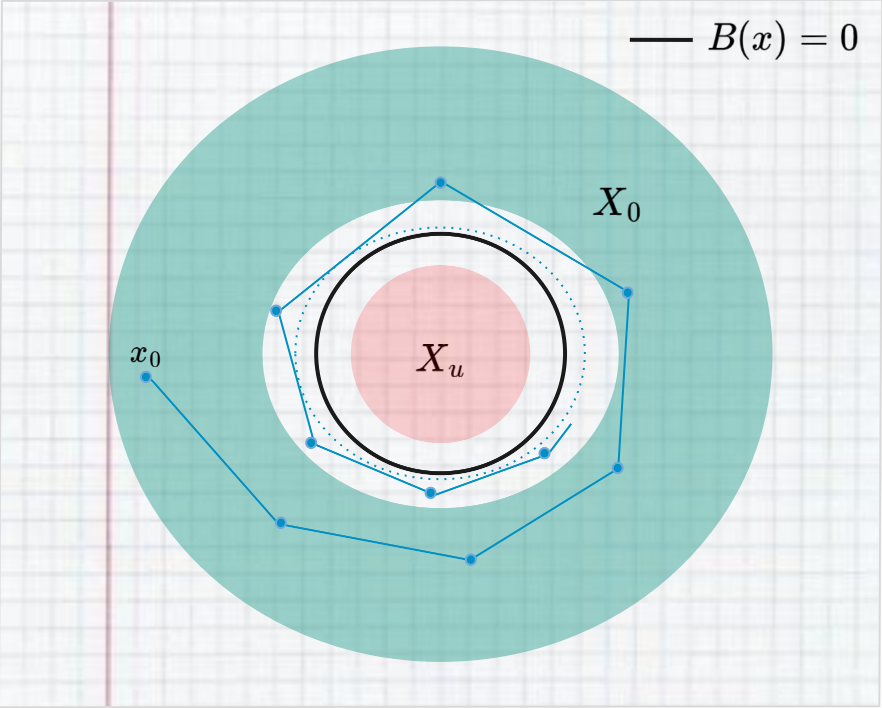

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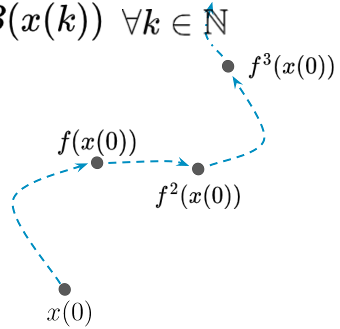

$$x(k+1) = f_{\sigma(k)}(x(k))$$

Generalization: $\exists B : \mathbb{R}^n \to \mathbb{R}$ such that

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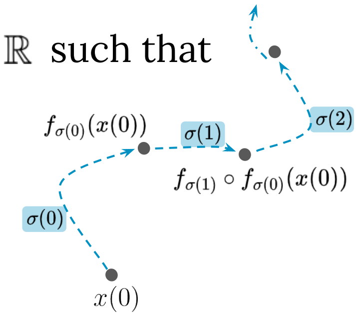
the initial state **and a switching signal**



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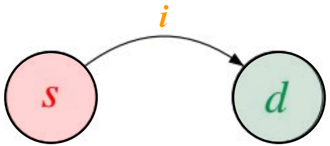

$$\forall x \in X, \forall (s, d, i) \in E, \quad B_d(f_i(x)) \leq B_s(x)$$

M.Anand, R. Jungers, M. Zamani & F. Allgöwer. Path-Complete Barrier Functions for Safety of Switched Linear Systems, CDC 2024.

M.Anand, R. Jungers, M. Zamani & F. Allgöwer. On the completeness and ordering of path-complete barrier functions, preprint, 2025

Example. Let us consider the 2-dimensional switched system with 2 modes

$$A_1 := \begin{bmatrix} 0.7 & 0.77 \\ -0.49 & 0.84 \end{bmatrix} \text{ and } A_2 := \begin{bmatrix} 0.7 & 0.77 \\ -0.49 & 0.56 \end{bmatrix}$$

with $X_0 = \{x_1, x_2 \mid {x_1}^2 + {x_2}^2 \leq 16\}$ and,

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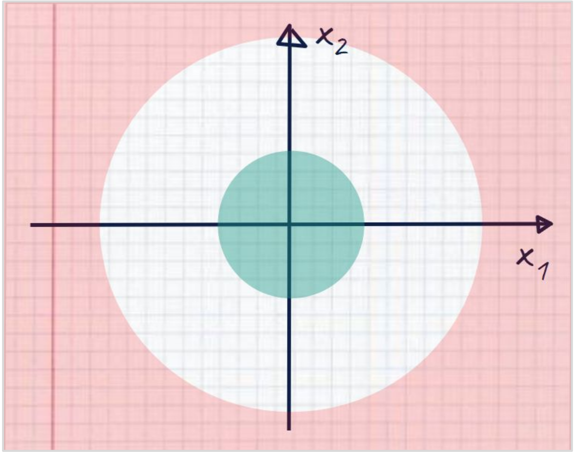

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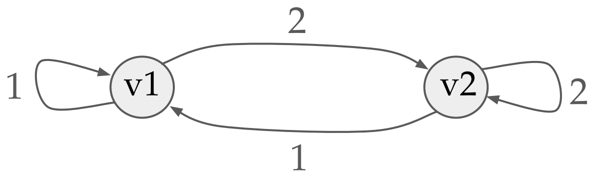

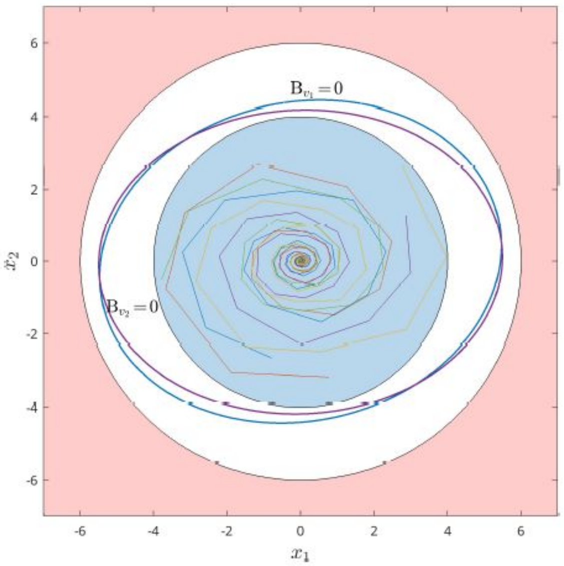

Illustration of PCBF. The 0 level sets of the pieces of the PCBF Bv1 and Bv2 corresponding to nodes v1 and v2 in G1 are labeled accordingly. The plot also shows some sample trajectories starting from X0 under arbitrary switching sequences, indicating that the system is safe.

M.Anand, R. Jungers, M. Zamani & F. Allgöwer. Path-Complete Barrier Functions for Safety of Switched Linear Systems, CDC 2024.

Feedback stabilization problem: Given a linear switched control system

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Can we design a feedback control stabilizing policy no matter the underlying switching rule?

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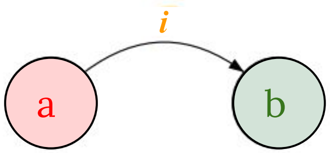
If there exist

$$\forall e = (a, b, i) \in E \qquad (A_i + B_i K_a)^\top P_b (A_i + B_i K_a) - P_a \prec 0$$



then $\phi(x) := K_{\gamma(x)}x$ robustly exponentially stabilizes the system with $\gamma(x) \in argmin_{s \in S}\{x^\top P_s x\}$

robust feedback controller
when the switching signal is unobservable

complete path-complete graph

mode-dependent feedback controllers
when the switching signal is observable

complete and deterministic path-complete graph

M. Della Rossa, T. Alves Lima, M. Jungers and R.M. Jungers. Graph-based conditions for feedback stabilization of switched and LPV systems, 2024

Conclusion

The path-complete formalism is a **unified** and **efficien**t approach to tackle the analysis of switched systems which formally describe how different **pieces** must be **related/composed** so that, **together**, they provide a **certificate**.

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Thanks for listening !

And feel free to ask any question

virginie.debauche@cs.ox.ac.uk

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