There Is No Band

Double Categories, Fragmented Spacetime, and an AQFT

Khyathi Komalan

California Institute of Technology

Topos Oxford Seminar, 2025

Contents/Structure of the Talk

- Constructing the Double Category Mink
- Constructing the Double Category **vNA**
- The Functor F : Mink $\to$ **vNA**
- Gluing in our AQFT
- Locality in our AQFT
- Summary
- Future Work
- References

Most constructions are paired with a scene from Mulholland Drive to illuminate the idea

There Was An Accident

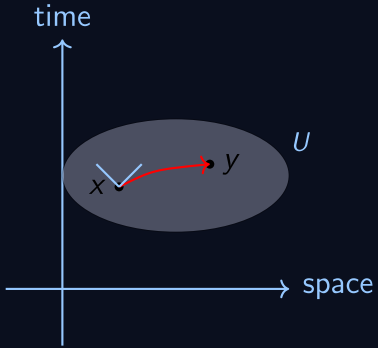
Fragments of a Theory

- We never observe all of spacetime.
- Like the woman, we wake up inside a fragment — a patch of reality with no access to the global picture.
- A physical theory must work locally — assigning structure to small, coherent regions.
- Ideally, we should be able to compose them, this means different such fragments can exist without contradiction.
- In our own "local space" or "fragment" we have our own ideas of morality, survival instinct, sense of time, etc.

Waking Up with Amnesia at a Stranger's Apartment

Objects: Causally Convex Regions

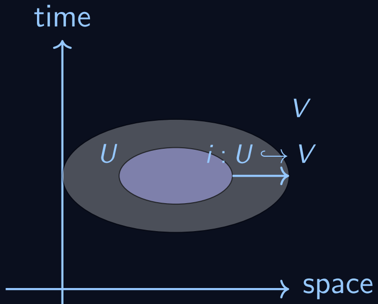
- Rita hides in the apartment. No signal escapes. No memory enters.
- We want regions like that: causally sealed fragments of spacetime.
- A region $U \subset \mathbb{R}^{1,1}$ is **causally convex** if:
  - For all $x, y \in U$, every causal curve from x to y lies entirely in U.
- These will be the **objects** in our double category Mink.



Betty Steps In

Vertical Morphisms: Looking Beyond the Fragment

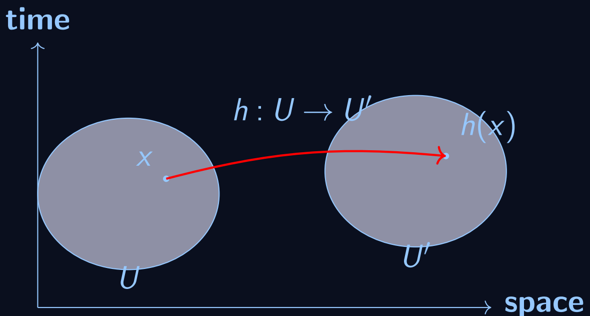
- Rita's life in her apartment sits inside bigger things — Betty's life, the city, the storyline, etc.
- This embedding is modelled by an **inclusion** of regions
- A vertical morphism in Mink is a map $i : U \hookrightarrow V$ between causally convex open subsets.
- Inclusions preserve causal structure and allow us to zoom out.



From Fragments to Flow

Horizontal Morphisms: Causal Flow Across Fragments

- Like Betty and Rita leaving the apartment and interacting with each other, causal flow connects fragments without containment.
- A **causal map** $h : U \to U'$ satisfies $x \leq y \Rightarrow h(x) \leq h(y)$.
- These are the **horizontal morphisms** of our double category — time-respecting relations between regions.



Winkie's Diner

Squares (2-morphisms): Two Paths to Terror

$$
\begin{array}{ccc}
U \text{ (dream/fear)} & \xrightarrow{h\text{:dream evolves}} & U' \text{ (imagination/threat)} \\
\Big\downarrow {\scriptstyle i\text{:inclusion}} & & \Big\downarrow {\scriptstyle i'\text{:inclusion}} \\
V \text{ (at winkie's diner)} & \xrightarrow[k\text{:surroundings evolve}]{} & V' \text{ (figure appears at winkie's alley}
\end{array}
$$

Our Double Category Mink

- Objects: causally convex regions
- Vertical Morphisms: inclusions of regions
- Horizontal Morphisms: causal map across regions
- Squares: commuting diagrams of inclusions and causal maps

Mr. Roque

Objects: von Neumann Algebras

- In **vNA**, objects are von Neumann algebras $\mathscr{M} \subseteq \mathscr{B}(H)$.
- They encode all observables for a given spacetime region.
- Like Mr. Roque's phone network: every possible "line" of action or measurement in his domain runs through a single, closed system.
- Complete, closed under composition, and the sole control hub for that fragment of reality.
- In our setting, these will often be *factors*, representing indivisible decision-making contexts with trivial center $Z(\mathscr{M}) = \mathbb{C} \cdot 1$.

Castigliane Brothers

Horizontal Morphisms: Hilbert Bimodules

- In **vNA**, horizontal morphisms are *Hilbert* $\mathscr{M}$–$\mathscr{N}$ *bimodules*.
- A Hilbert space H with commuting normal $*$-representations: $\pi_L : \mathscr{M} \to \mathscr{B}(H)$, $\pi_R : \mathscr{N}^{op} \to \mathscr{B}(H)$.
- They "translate" between two algebras, implementing transformations while respecting both left and right actions.
- A forced recasting bridges Adam's vision (left action) and the imposed image (right action), preserving the film's structure but changing its representation.
- Just as a bimodule must respect both source and target algebras, Adam must now work within both his original framework and the brothers' imposed choice.

Mr. Roque Talking to Mr. Ray

Vertical Morphisms: Normal Unital $*$-Homos

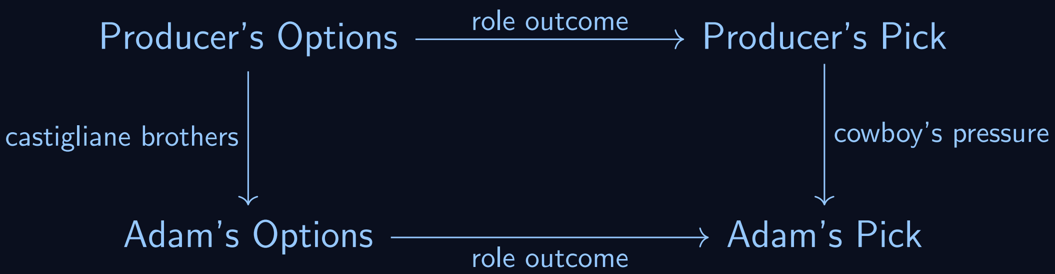
- In **vNA**, vertical morphisms are *normal unital $*$-homomorphisms* $\phi : \mathscr{M} \to \mathscr{N}$.
- $*$-homomorphism: preserves multiplication and adjoints: $\phi(ab) = \phi(a)\phi(b)$, $\phi(a^*) = \phi(a)^*$.
- Unital: $\phi(\mathbf{1}_{\mathscr{M}}) = \mathbf{1}_{\mathscr{N}}$.
- Normal: continuous in the ultraweak topology; preserves suprema of increasing nets of projections.
- Roque's instructions to Ray pass intact — structure, identity, and all operational rules remain unchanged.

The Cowboy

Squares (2-morphisms): Intertwiners

- The Cowboy enforces “This is the girl” — no matter if the role is locked before or after his pressure, the result is identical.
- In **vNA**, squares are *intertwiners* between Hilbert bimodules, ensuring two paths of action coincide.
- An intertwiner $T : E \to F$ satisfies $T(\pi_L(a)\,\xi\,\pi_R(b)) = \pi'_L(a)\,T(\xi)\,\pi'_R(b)$ for all a, b.
- Here: “Cowboy pressure then lock” $=$ “lock then Cowboy pressure” — both yield the same casting outcome.

Cowboy Scene: Intertwiner



Confusion at Winkie's Diner

Vertical Composition: Diagram

- $\mathscr{M}$ = knowledge algebra before the name "Diane" appears
- ϕ = processing the nametag clue
- $\mathscr{N}$ = knowledge algebra including the name "Diane Selwyn" as a hypothesis
- ψ = decision to check the phonebook
- $\mathscr{P}$ = knowledge algebra with the address to visit

$$\mathscr{M} \xrightarrow{\quad\phi\quad} \mathscr{N} \xrightarrow{\quad\psi\quad} \mathscr{P}$$

Vertical Composition: Composing *-Homos

- Rita suspects she might be "Diane", leading to address lookup, then visit to her apartment.
- Each step is a normal unital $*$-homomorphism within the same spacetime region/algebraic context.
- Vertical composition $=$ chaining these maps: $(\psi \circ \phi) : \mathscr{M} \to \mathscr{P}$, preserving multiplication, adjoints, the unit, and normality.
- Narratively: suspicion $\to$ name $\to$ address $\to$ apartment — each refines the prior state without leaving the original regional context.

Vertical Composition: Normal and Unital

- **Unital:** The "unit" (baseline facts + ultimate goal of uncovering Rita's identity) remains fixed across all knowledge states.
- Refining from nametag → phonebook → address never changes *what* we are trying to solve.
- **Normal:** Updates respect limits of increasing information — assembling partial clues leads smoothly to the final state.
- Smaller observations about Diane (name tag, address hints) accumulate without breaking the logic of the search.

Arriving at "Diane"'s Apartment

Horizontal Composition: Diagram

- C: vNA for public knowledge at Winkie's.
- A: vNA for the address info.
- B: vNA for Diane's apartment.

$$C \xrightarrow{\text{phonebook}} A \xrightarrow{\text{travelling}} B$$

$$\xrightarrow{\quad\text{connes fusion over A}\quad}$$

$$C \xrightarrow{\text{phonebook-to-door}} B$$

Horizontal Composition: Connes Fusion

- Waiter clue $\to$ address $\to$ Diane's door — two influence paths merge.
- Horizontal 1-cells: Hilbert bimodules ${}_{\mathscr{C}}E_{\mathscr{A}}$ and ${}_{\mathscr{A}}F_{\mathscr{B}}$.
- **Fusion:** $E \boxtimes_{\mathscr{A}} F$ balances over $\mathscr{A}$ and completes to a Hilbert $\mathscr{C}$–$\mathscr{B}$ bimodule.
- **Meaning:** Two cross-context channels become one, preserving left $\mathscr{C}$ and right $\mathscr{B}$ actions.

Our Double Category **vNA**

- Objects: von Neumann Algebras (factors)
- Vertical Morphisms: normal unital *-homomorphisms
- Horizontal Morphisms: Hilbert bimodules
- Squares: Intertwiners
- Vertical Composition: composition of normal unital *-homomorphisms
- Horizontal Composition: connes fusion

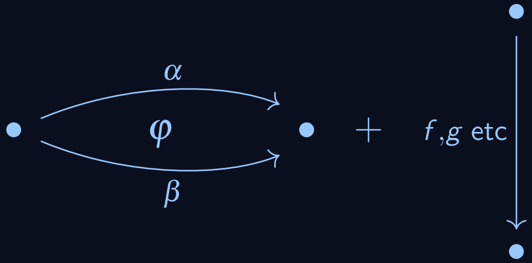
There Is No Band

Club Silencio: Globularly Generated Double Cats

- We'll build double categories from the least data, then "project" to the real stage.
- Defns: decorated bicategory, internalization, globular generation, and the free lift.
- Then: why Mink and **vNA** fit this scheme.

Decorated Bicategory

- A pair $(\mathscr{B}^*, \mathscr{B})$ with the *same objects*.
- $\mathscr{B}^*$: a **category** of vertical arrows (no 2-cells).
- $\mathscr{B}$: a **bicategory** of horizontal 1-cells and 2-cells.
- Idea: list the objects, the "up/down" arrows, and the "left/right + 2-cells".



From a Double Category Back to Data

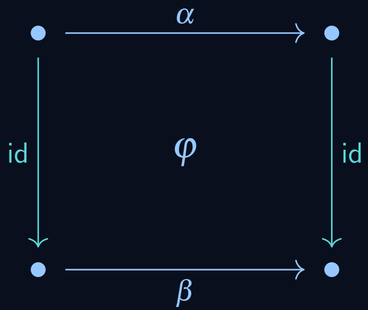
- Given a double category $\mathscr{C}$, its **decorated horizontalization** is $H^*\mathscr{C} = (\mathscr{C}_0, H\mathscr{C})$.
- $\mathscr{C}_0$: objects + vertical arrows (forget squares).
- $H\mathscr{C}$: objects + horizontal 1-cells + their 2-cells (forget vertical composites).
- We "mic the band": keep tracks (data) but hide the stage machinery.

Internalization Problem

- Input: a decorated bicategory $(\mathscr{B}^*, \mathscr{B})$.
- Task: build a double category $\mathscr{C}$ whose visible data is exactly that input: $H^*\mathscr{C} = (\mathscr{B}^*, \mathscr{B})$.
- Interpretation: can we realize the "recording" as a full "stage show"?

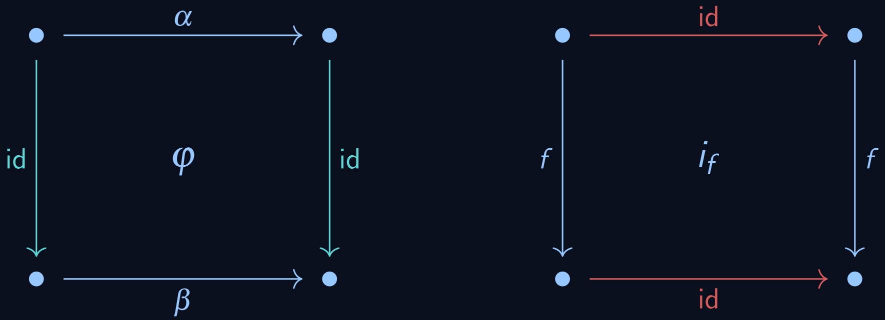
Globular Squares and Generation

- A **globular** square: vertical edges are identities; only horizontal data varies.
- $\mathscr{C}$ is **globularly generated** if every square is built by vertical/horizontal composition from globular ones.
- Notation: $\gamma\mathscr{C} =$ sub-double category generated by globular squares



Free Globularly Generated Double Category

- From $(\mathscr{B}^*, \mathscr{B})$ build $Q(\mathscr{B}^*, \mathscr{B})$: add only the squares *forced* by composition axioms.
- Universal: any realization $\mathscr{C}$ with $H^*\mathscr{C} = (\mathscr{B}^*, \mathscr{B})$ receives a unique strict double functor from Q.
- Horizontal/vertical compositions of:



Mink as Globularly Generated

- $\mathscr{B}^*$: inclusions of causally convex regions (vertical).
- $\mathscr{B}$: causal, orientation-preserving embeddings; 2-cells = commuting squares.
- Every square factors into globular pieces (whisker by inclusions) and composes back.
- Thus Mink $\cong \gamma\mathscr{C}$ for its standard square-of-embeddings $\mathscr{C}$.

vNA (Factors) as Globularly Generated

- $\mathscr{B}^*$: normal unital $*$-homomorphisms (vertical).
- $\mathscr{B}$: Hilbert bimodules as 1-cells; intertwiners as 2-cells.
- Globular squares $=$ intertwiners with identity verticals; whiskering by $*$-homs yields all squares.
- Free lift Q projects onto the standard "linear" double category (Connes fusion respected).

Silencio: No Band, Just a Projection

- Free track Q exists from minimal data $(\mathscr{B}^*, \mathscr{B})$.
- Your concrete stage $\mathscr{C}$ is a canonical projection of Q.
- All the "music" (squares) is synthesized from globular stems.

The Blue Box

The AQFT Double Functor

- The box = double functor $F : \mathrm{Mink} \to \mathbf{vNA}$.
- Objects: $U \mapsto \mathscr{A}(U)$ (von Neumann algebra).
- Vertical: inclusions $\mapsto$ normal unital $*$-homs.
- Horizontal: causal maps $\mapsto$ Hilbert bimodules; compose via Connes fusion.
- 2-cells: squares $\mapsto$ intertwiners, coherence via interchange.

Analogy: opening the box forces fragments (Mink pieces) to collapse into a single algebraic translation rule.

Inside the Box: An AQFT Net

- **Isotony:** $U \subset V \Rightarrow \mathscr{A}(U) \subset \mathscr{A}(V)$.
- **Locality:** spacelike $U, V \Rightarrow [\mathscr{A}(U), \mathscr{A}(V)] = 0$.
- **Covariance:** symmetries act by natural isomorphisms.
- **Time-slice:** if U contains a Cauchy surface for V, then $\mathscr{A}(U) = \mathscr{A}(V)$.

Analogy: opening the box doesn't reveal "reality" — it collapses the dream into a single, inevitable structure: the AQFT net.

Wake Up

Gluing in the Storyline

Fragment A
Cowboy (smile)
“Time to wake up”

→ match on shared anchor →

Shared Anchor
doorway frame
same line / room tone

→ match on shared anchor →

Fragment B
Cowboy (unsmiling)
door closes

⇓ keep continuity across the joins

Glued Sequence
one coherent wake–up beat

Gluing in our AQFT

- **Data:** a net $A : \mathscr{O}(M) \to \mathbf{vNA}$ from $F : \mathrm{Mink} \to \mathbf{vNA}$.
- **Rule to keep:** if $U \perp V \subset T$, then inside $A(T)$ the images of $A(U)$ and $A(V)$ commute.

Recipe (3 steps):

1. Match on overlaps (Čech compatibility).
2. Enforce independence: $[A(U), A(V)] = 0$ for $U \perp V$.
3. Take the universal algebra $A(M)$ with 1–2.

$$A(M) \;\cong\; \big(\mathrm{colim}_{\check{C}(U_\alpha)} A\big) / \langle\langle \text{overlap equalities, disjoint-commutation} \rangle\rangle$$

(Cowboy "wake up": we don't just splice pieces; we enforce the rule wherever they meet.)

Gluing in our AQFT (Diagram)

Two patches: $M = U \cup V,\ W = U \cap V$

$$\begin{array}{ccc} A(W) & \xrightarrow{i_U} & A(U) \\ \big\downarrow i_V & & \big\downarrow \\ A(V) & \rightarrow & \boxed{P} \end{array} \qquad P := A(U) \underset{A(W)}{*} A(V)$$

$$\boxed{P} \twoheadrightarrow \boxed{A(U \cup V)} \quad \text{impose} \quad [A(U), A(V)] = 0 \ \text{ if } U \perp V$$

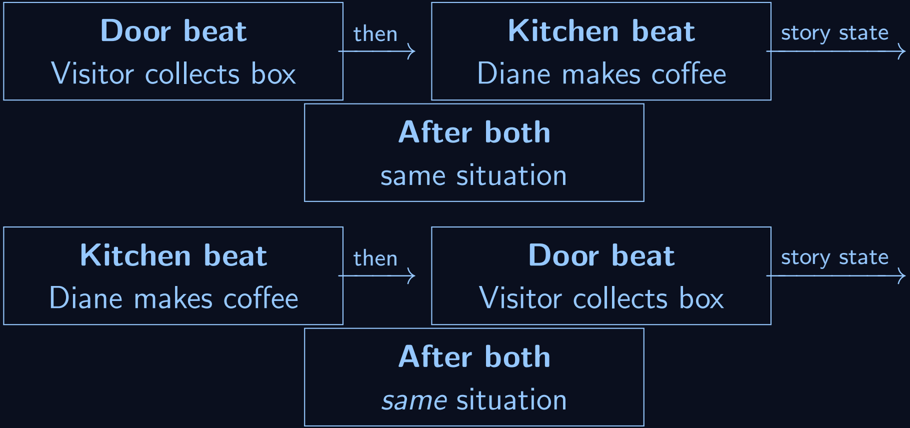
$$\textit{Universal:} \quad \begin{cases} j_U : A(U) \to B,\ j_V : A(V) \to B, \\ j_U \circ i_U = j_V \circ i_V,\ [j_U(A(U)), j_V(A(V))] = 0 \end{cases} \Rightarrow \exists!\ \phi : \boxed{A(U \cup V)} \to B.$$

Read: (1) align on the overlap, (2) enforce "independent parts commute," (3) obtain the unique global algebra receiving all compatible, commuting maps.

The Neighbour

Diane Making Coffee

Locality in the Storyline



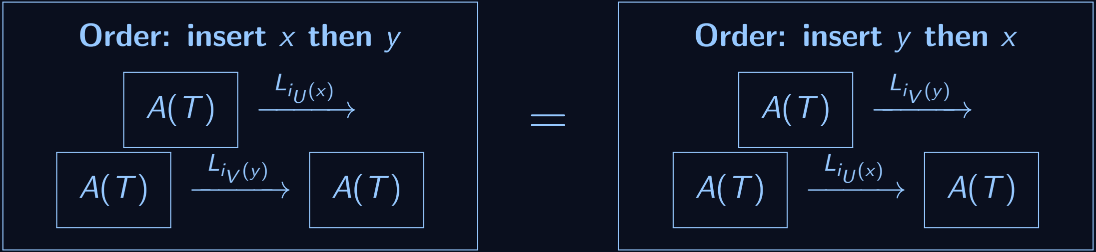
Order doesn't change the meaning *when beats are independent* (spacelike).

Locality in our AQFT

- **Assume:** $U \perp V \subset T$ (spacelike/disjoint inside T).
- **Claim:** inside $A(T)$, the images of $A(U)$ and $A(V)$ commute.
- **Model:** $F : \mathrm{Mink} \to \mathbf{vNA}$ is a double functor.
- Vertical $=$ inclusions $\Rightarrow$ $*$-homs; horizontal $=$ propagation as Hilbert bimodules (Connes fusion).
- **Why:** the *interchange law* makes the two pastings (apply U then V vs V then U) equal $\Rightarrow$ commuting actions.

Locality in our AQFT (Diagram)

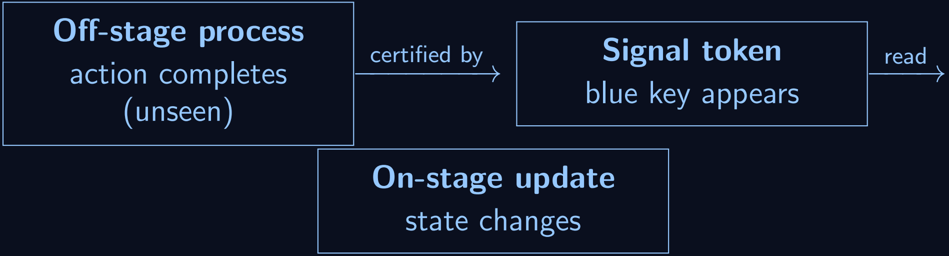
Fix elements $x \in A(U),\ y \in A(V)$.



Equality of the two composites is equivalent to $i_U(x)\, i_V(y) = i_V(y)\, i_U(x)$ inside $A(T)$.

The Blue Key

Why Double Cats? Signals/Channels

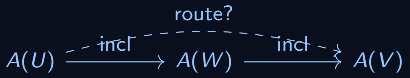


- **1-cat limit:** only inclusions $\Rightarrow$ $*$-homs; no place for *channels/CP maps*.
- **Double win:** horizontal 1-cells = correspondences (Hilbert bimodules) model *CP maps*; composition = Connes fusion.
- **Concrete:** $\Phi_2 \circ \Phi_1$ corresponds to $M_1 \otimes_B M_2$. Signals as small on-stage effects after off-stage processes.

Shortcut

Why Double Cats? Transport

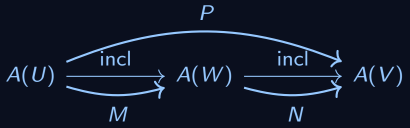
1-cat net



only inclusions ($*$-homs)

no notion of a route

Double AQFT



horizontals = bimodules/channels

fusion: $P \cong M \otimes_{A(W)} N$

Secret path vs direct walk: 1-cat sees only places; the double adds routes and coherently equates them.

Dinner Party

Why Double Cats? Interfaces & Morita

1-cat net: compare by natural transformation

$$\begin{array}{ccc} \mathscr{A}(U) & \xrightarrow{\mathscr{A}(i_{U,V})} & \mathscr{A}(V) \\ \eta_U \big\downarrow & \eta_V \circ \mathscr{A}(i_{U,V}) = \mathscr{B}(i_{U,V}) \circ \eta_U & \big\downarrow \eta_V \\ \mathscr{B}(U) & \xrightarrow{\mathscr{B}(i_{U,V})} & \mathscr{B}(V) \end{array}$$

Strict comparison by $*$-homs only; no boundary/defect notion.

Double net: compare by interfaces

$$\begin{array}{ccc} \mathscr{A}(U) & \xrightarrow{\mathscr{A}(i_{U,V})} & \mathscr{A}(V) \\ M_U \big| & \mathscr{A}(i_{U,V}) \triangleright M_U \cong M_V \triangleleft \mathscr{B}(i_{U,V}) & \big| M_V \\ \mathscr{B}(U) & \xrightarrow{\mathscr{B}(i_{U,V})} & \mathscr{B}(V) \end{array}$$

Interfaces = bimodules, compose by fusion $M \otimes_{\mathscr{B}(\cdot)} N$; invertible $\Rightarrow$ Morita equivalence.

Summary

- Double functor $F : \mathrm{Mink} \to \mathbf{vNA}$: vertical $*$-homs, horizontal bimodules, intertwiners.
- Local net $A : \mathscr{O}(M) \to \mathbf{vNA}$; isotony/covariance/time-slice as standard.
- Locality: interchange $\Rightarrow$ commuting subalgebras for $U \perp V$.
- Gluing: rule-preserving colimits; global net from pieces.
- New: channels (signals), routes (transport), interfaces/Morita — not available in 1-cat nets.

Future Work

- **Foundations:** prove gluing as an operadic left Kan extension; locality/time-slice from interchange.
- **Examples:** free scalar/Dirac nets; compute glued $\mathscr{A}(M)$; check Haag duality, split property.
- **Computational bridge:** link to quantum λ-calculus (vNAs already model it in ∞-dim.); evaluation as fusion.
- **Interfaces:** boundary/defect gluing; classify invertible bimodules (Morita).
- **Type 3 vNAs:** how we can use this AQFT definition to learn more about Type 3 vNAs

References I

Juan Orendain.
Free Globularly Generated Double Categories I
Theory and Applications of Categories, Vol. 34, No. 42, 2019, pp. 1343-1385.

Juan Orendain.
Globularly Generated Double Categories II: The Canonical Double Projection
arXiV, 2021

Uffe Haagerup.
The Standard Form of Von Neumann Algebras
Mathematica Scandinavica, Vol. 37, No. 2, pp. 271-283, 1975

Romeo Brunetti, Klaus Fredenhagen.
Algebraic approach to Quantum Field Theory
arXiV, 2004

References II

Urs Schreiber.
AQFT from n–functorial AQFT
Communications in Mathematical Physics, Springer Science and Business Media LLC, Vol. 291, No. 2, pp. 357-401, 2009

Angelos Anastopoulos, Marco Benini.
Gluing algebraic quantum field theories on manifolds
Annales Henri Poincaré, Springer Science and Business Media LLC, 2025

Alan Shaw.
Scene-by-Scene Analysis of Mulholland Drive
mulholland-drive.net, unknown year