Double functorial representation of indexed monoidal structures

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Plan

1. Classical hyperdoctrines
2. Loosen up
3. Double (pseudo)functors from doctrines
4. Doctrines from double functors
5. Logic as a system
6. Food for thought

Classical hyperdoctrines

Start with

1. a category $\mathcal{C}$ of "things you want to talk about";
2. associate to each object $X \in \mathcal{C}$ a collection of "facts" that might be true of elements of type X;
3. explain how you do logical operations.

Lawvere's perspective on logic: *quantifiers are adjoints*.

Classical hyperdoctrines

Definition

A regular hyperdoctrine is a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ such that:

1. Each poset PX is $\wedge$-semilattice;
2. For each morphism $f\colon X \to Y$ in $\mathcal{C}$, the functor $Pf\colon PY \to PX$ has a left adjoint $\exists f$;
3. These adjoints satisfy the Beck-Chevalley condition: for any pullback square
$$\begin{array}{ccc} A & \xrightarrow{h} & I \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & J \end{array}$$, the canonical map $\exists h \circ Pk \Rightarrow Pg \circ \exists f$ is invertible;
4. These adjoints satisfy Frobenius reciprocity: for each $f\colon X \to Y$, the canonical map $\exists f(Pf \wedge \mathrm{id}_{PX}) \Rightarrow \mathrm{id}_{PY} \wedge \exists f$ is invertible.

Examples

1. $\mathcal{C} = \text{Set}$, and P is the powerset functor:

• Predicates about X are subsets of X (“the subset of elements satisfying the predicate"). In particular, the predicates about 1 are truth-values $\top$ and $\bot$;
• Given a map $f\colon Y \to X$, Pf maps $\{y \in Y\colon \varphi(y)\}$, to $\{x \in X\colon \varphi(f(x))\}$;
• $\exists f$ maps $\{x \in X\colon \varphi(x)\}$ to its image $\{y \in Y\colon \exists x \in X\colon y = f(x) \wedge \varphi(x)\}$ under f.

Note that if $\varphi(x, y)$ has two variables and $\pi_Y\colon X \times Y \to Y$ is a projection map, then $\exists\pi_Y(\{(x, y)\colon \varphi(x, y)\}) = \{y \in Y\colon \exists x.\ \varphi(x, y)\}$ recovers the usual quantification over a variable.

• Beck-Chevalley guarantees that you can substitute y in $\varphi(x, y)$ for some t before or after quantifying over x and get the same result.

• Similarly, Frobenius tells you that if x isn't free in φ, then $\exists x.\ (\varphi(y) \wedge \psi(x, y))$ is equivalent to $\varphi(y) \wedge \exists x.\ \psi(x, y)$.

Examples

2. If T is a regular theory (i.e., its axioms uses only $\wedge, \top$, and existential quantification), then we can take $\mathcal{C}$ to be its *syntactic category*.

To each context Γ, we associate its Lindenbaum-Tarski algebra: formulas on those variables, quotiented out by the relation saying $\varphi(\bar{x}) \equiv \psi(\bar{y})$ if $T \vdash_{\text{reg}} \varphi(\bar{x}) \iff \psi(\bar{x})$.

This lets us see theories such as the theory of graphs as regular hyperdoctrines.

Loosen up

This can be slightly too restrictive:

1. There is no reason for the predicates to be elements of a poset; maybe they are (e.g.) objects of a category
2. There is no reason why your $\wedge$ operation must be a categorical product of predicates; maybe it is a tensor product of some kind
3. There is no reason to insist that you can quantify over any morphism; maybe you just need some.

If $(\mathcal{C}, L, R)$ is a cartesian adequate triple and the predicates form a (pseudo)monoid in a cartesian 2-category $\mathfrak{K}$, we call the analogous notion (where we can only quantify over maps in L) a $(\mathcal{C}, L, R)$-regular indexed monoidal structure in $\mathfrak{K}$.

Examples

1. If $\mathcal{C}$ is a category with finite limits, then we can define a pseudofunctor $P\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{SM}(\mathsf{Cat})_{\mathsf{strong}}$ by $X \mapsto \mathcal{C}/X$ (which is cartesian monoidal), where $Pf : PY \to PX$ is given by pulling back along f.

Each such Pf has a left adjoint $\Sigma(f)$ (the *dependent sum*), and the fact that these adjunctions satisfy the Beck-Chevalley and Frobenius properties is given by the well-known pullback pasting lemma.

Examples

2. For each set A, define $PA := [0, \infty]^A$, which is a monoidal poset under $+$ and $\geq$.

For each function $f\colon A \to B$, precomposing predicates with f yields an order-preserving map $Pf\colon PB \to PA$ (which is strict monoidal), and the assignment $f \mapsto Pf$ is functorial.

Each such Pf admits a left adjoint, namely $\exists f\colon PA \to PB$ taking a predicate φ over A to the predicate $b \mapsto \inf_{a \in f^{-1}(b)}(\varphi(a))$ over B.

For more on this logic, check [2].

Theorem

The following data are equivalent for a cartesian adequate triple $(\mathcal{C}, L, R)$ *and cartesian 2-category* $\mathfrak{K}$*:*

1. *A* $(\mathcal{C}, L, R)$*- regular indexed monoidal structure* $P\colon \mathcal{C} \to \boldsymbol{SM}(\mathfrak{K})_{strong}$ *in* $\mathfrak{K}$*;*
2. *A lax symmetric monoidal (strong) double pseudofunctor* $P^{\bullet}\colon \mathbb{S}pan(\mathcal{C}, L, R)^{\mathrm{op}} \to \mathbb{Q}t(\mathfrak{K})$ *whose monoidal laxitors are companion commuter cells.*

Double pseudofunctors from regular indexed structures

Fix a $(\mathcal{C}, L, R)$- regular indexed monoidal structure $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{SM}(\mathfrak{K})_{\mathsf{strong}}$. Then:

- The left adjoints $\exists f \vdash Pf$ for $f \in L$ are also pseudofunctors, with the structure induced from Pf by taking mates;
- The monoidal structure can be moved from the fibres to P itself, i.e., $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{SM}(\mathfrak{K})_{\mathsf{strong}}$ corresponds to a (cartesian) lax symmetric monoidal $P\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{K}$ (c.f. [6] and [7]).

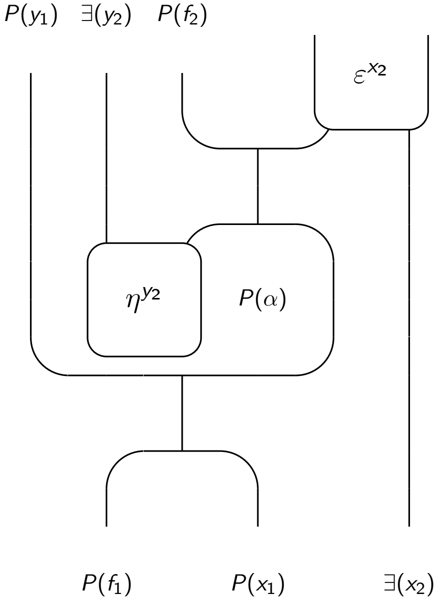
Double pseudofunctors from regular indexed structures

We extend $P\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{K}$ to a double pseudofunctor $P^\bullet\colon \mathbb{S}pan(\mathcal{C}, L, R)^{\mathrm{op}} \to \mathbb{Q}t(\mathfrak{K})$.

The loose component of $P^\bullet$ maps a span $X_1 \xleftarrow{x_1} X \xrightarrow{x_2} X_2$ with $x_2 \in L$ to $\exists x_2 \circ P x_1$ and a square

$$
\begin{tikzcd}
Y_1 \arrow[r, "Y", "\shortmid"{marking}] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & Y_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "X"', "\shortmid"{marking}] & X_2
\end{tikzcd}
$$

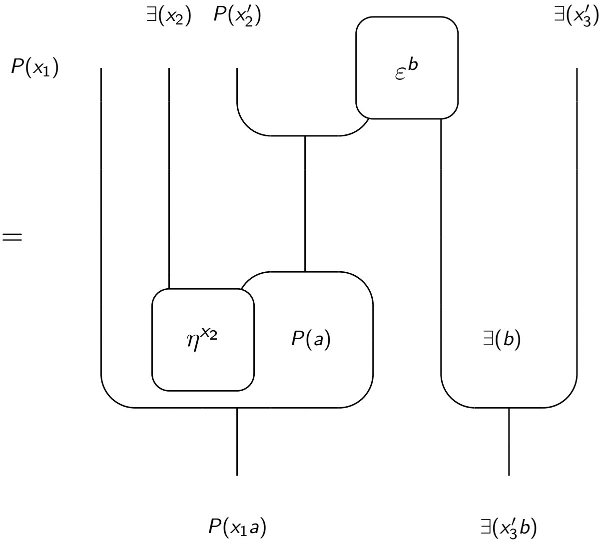
to


$P(y_1)$
$\exists(y_2)$
$P(f_2)$
ε^{x_2}
η^{y_2}
$P(\alpha)$
$P(f_1)$
$P(x_1)$
$\exists(x_2)$

This comes with a loose compositor whose component $P^\bullet(X \odot X') \Rightarrow P^\bullet(X) \odot P^\bullet(X')$

at a composite span
$$\begin{tikzcd}
X \odot X' \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "x_1"] \arrow[d, "x_2"] & X_1 \\
X' \arrow[r, "x_2'"'] \arrow[d, "x_3'"'] & X_2 & \\
X_3 & &
\end{tikzcd}$$
is given by

$$(P^{\bullet}_{\odot})_{x,x'} \coloneqq$$



This is a pseudonatural transformation, whose inverse cell is given by

$$
\begin{tikzcd}
PX_1 \arrow[rr, "P(x_1 a)", "\shortmid" marking] \arrow[d, equal] & & P(X \odot X') \arrow[rr, "\exists(x_3' b)", "\shortmid" marking] \arrow[d, equal] & & PX_3 \arrow[d, equal] \\
PX_1 \arrow[r, "Px_1"', "\shortmid" marking] \arrow[d, equal] & PX \arrow[r, "Pa"', "\shortmid" marking] \arrow[d, equal] & P(X \odot X') \arrow[r, "\exists b"', "\shortmid" marking] & PX' \arrow[r, "\exists x_3'"', "\shortmid" marking] \arrow[d, equal] & PX_3 \arrow[d, equal] \\
PX_1 \arrow[r, "Px_1"', "\shortmid" marking] & PX \arrow[r, "\exists x_2"', "\shortmid" marking] & PX_2 \arrow[r, "Px_2'"', "\shortmid" marking] & PX' \arrow[r, "\exists x_3'"', "\shortmid" marking] & PX_3
\arrow[from=1-1, to=2-3, phantom, "\Uparrow"]
\arrow[from=1-3, to=2-5, phantom, "\Uparrow"]
\arrow[from=2-2, to=3-4, phantom, "\mathcal{B}^{a,b}_{x_2',x_2} \Uparrow"]
\end{tikzcd},
$$

where $\mathcal{B}^{a,b}_{x_2',x_2}$ is a Beck-Chevalley cell.

The unitor $P^\bullet_e \colon e \circ P \to P^\bullet_1 \circ e$ is the pseudonatural isomorphism with components $e_{PX} \to P^\bullet_1(e_X)$ given by

$$\begin{tikzcd} PX \arrow[r, "P(\mathrm{id}_X)"] \arrow[d, equal] & PX \arrow[r, "\exists(\mathrm{id}_X)"] \arrow[d, equal] & PX \arrow[d, equal] \\ PX \arrow[r, equal] \arrow[ur, phantom, "\Uparrow \iota^P_X"] & PX \arrow[r, equal] \arrow[ur, phantom, "\Uparrow (\iota^\exists_X)^{-1}"] & PX \end{tikzcd} .$$

Double pseudofunctors from regular indexed structures

Note that the Beck-Chevalley property was used to ensure pseudonaturality of the loose compositors, making $P^{\bullet}$ a double pseudofunctor.

Frobenius reciprocity is linked to the monoidal structure.

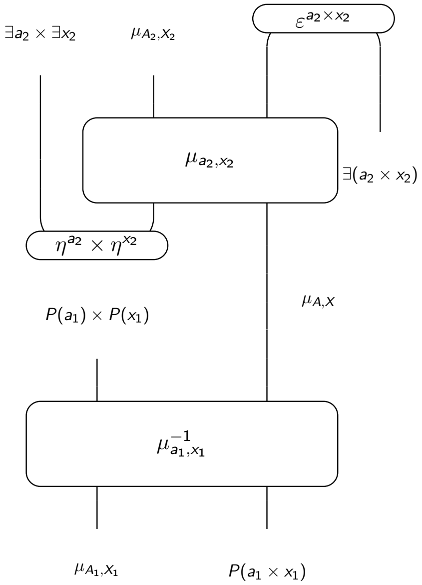
Double pseudofunctors from regular indexed structures

Explicitly, the cartesian monoidal structure on $P \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{K}$ has:

- A monoidal laxitor $\mu_{X,Y} \colon PX \times PY \to P(X \times Y)$ given by the composite $PX \times PY \xrightarrow{P(\pi_X) \times P(\pi_Y)} P(X \times Y)^2 \xrightarrow{\otimes_{X \times Y}} P(X \times Y)$.
- A monoidal unit similarly induced by the unit of the pseudomonoid $P(1)$.

The monoidal laxitor μ extends to a tight pseudotransformation, whose tight component is μ and whose loose component $\mu^\bullet \colon P^\bullet(- \times -) \to P^\bullet \times P^\bullet$ is given by

$$\mu^{\bullet}_{A,X} :=$$

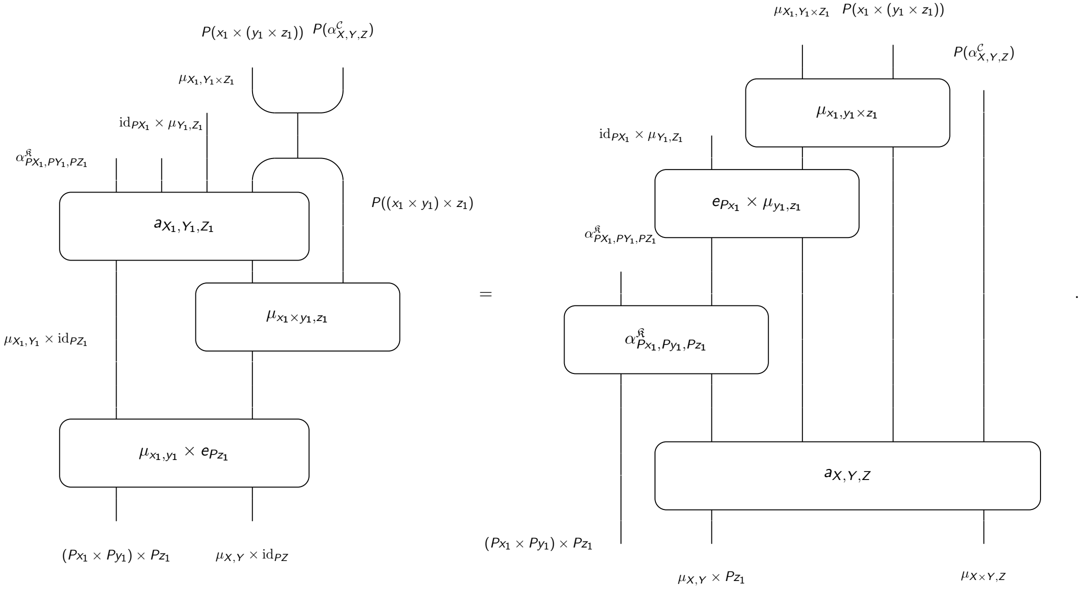


$\exists a_2 \times \exists x_2$
μ_{A_2,X_2}
$\varepsilon^{a_2 \times x_2}$
μ_{a_2,x_2}
$\exists(a_2 \times x_2)$
$\eta^{a_2} \times \eta^{x_2}$
$\mu_{A,X}$
$P(a_1) \times P(x_1)$
$\mu^{-1}_{a_1,x_1}$
μ_{A_1,X_1}
$P(a_1 \times x_1)$

This makes $P_1^\bullet$ symmetric comonoidal: since P is a lax symmetric monoidal pseudofunctor, there are invertible cells

$$\begin{tikzcd}
(PA \times PB) \times PC \arrow[r, "\alpha^{\mathfrak{K}}_{PA,PB,PC}"] \arrow[d, "\mu_{A,B} \times \mathrm{id}_{PC}"'] & PA \times (PB \times PC) \arrow[d, "\mathrm{id}_{PA} \times \mu_{B,C}"] \\
P(A \times B) \times PC \arrow[r, Rightarrow, "a_{A,B,C}"'] \arrow[d, "\mu_{A \times B,C}"'] & PA \times P(B \times C) \arrow[d, "\mu_{A,B \times C}"] \\
P((A \times B) \times C) \arrow[r, "P(\alpha^{\mathcal{C}}_{A,B,C})"'] & P(A \times (B \times C))
\end{tikzcd}$$

in $\mathfrak{K}$ such that



$P(x_1 \times (y_1 \times z_1))$
$P(\alpha^{\mathcal{C}}_{X,Y,Z})$
$\mu_{X_1, Y_1 \times Z_1}$
$\mathrm{id}_{PX_1} \times \mu_{Y_1, Z_1}$
$\alpha^{\mathfrak{K}}_{PX_1, PY_1, PZ_1}$
$P((x_1 \times y_1) \times z_1)$
a_{X_1, Y_1, Z_1}
$\mu_{x_1 \times y_1, z_1}$
$\mu_{X_1, Y_1} \times \mathrm{id}_{PZ_1}$
$\mu_{x_1, y_1} \times e_{Pz_1}$
$(Px_1 \times Py_1) \times Pz_1$
$\mu_{X,Y} \times \mathrm{id}_{PZ}$
$=$
$\mu_{X_1, Y_1 \times Z_1}$
$P(x_1 \times (y_1 \times z_1))$
$P(\alpha^{\mathcal{C}}_{X,Y,Z})$
$\mu_{x_1, y_1 \times z_1}$
$\mathrm{id}_{PX_1} \times \mu_{Y_1, Z_1}$
$e_{Px_1} \times \mu_{y_1, z_1}$
$\alpha^{\mathfrak{K}}_{PX_1, PY_1, PZ_1}$
$\alpha^{\mathfrak{K}}_{Px_1, Py_1, Pz_1}$
$a_{X,Y,Z}$
$(Px_1 \times Py_1) \times Pz_1$
$\mu_{X,Y} \times Pz_1$
$\mu_{X \times Y, Z}$

A dual equation holds for $\exists$, since it is comonoidal. We can put them together to show that for loose X, Y, Z, we have $\alpha_{P^\bullet X, P^\bullet Y, P^\bullet Z} \circ (\mu^\bullet_{X,Y} \times e_{P^\bullet Z}) \circ \mu^\bullet_{X \times Y, Z} = (e_{P^\bullet X} \times \mu^\bullet_{Y,Z}) \circ \mu^\bullet_{X, Y \times Z} \circ P^\bullet(\alpha^{\mathsf{Span}(\mathcal{C})}_{X,Y,Z})$.

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$$\alpha_{P^\bullet X, P^\bullet Y, P^\bullet Z} \circ (\mu^\bullet_{X,Y} \times e_{P^\bullet Z}) \circ \mu^\bullet_{X \times Y, Z} = (e_{P^\bullet X} \times \mu^\bullet_{Y,Z}) \circ \mu^\bullet_{X, Y \times Z} \circ P^\bullet(\alpha^{\mathsf{Span}(\mathcal{C})}_{X,Y,Z}).$$
A similar argument also works for the monoidal unit laws and the symmetry axioms.

Commuters

Definition (1)

A square
$$\begin{tikzcd} A_1 \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[d, "f_2"] \\ B_1 \arrow[r, "B"', "\shortmid"{marking}] & B_2 \end{tikzcd}$$
in a double category is a companion commuter if

$$\begin{tikzcd} A_1 \arrow[r, equal] \arrow[d, equal] & A_1 \arrow[r, "A", "\shortmid"{marking}] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[r, "f_2^{>}", "\shortmid"{marking}] \arrow[d, "f_2"] & B_2 \arrow[d, equal] \\ A_1 \arrow[r, "f_1^{>}"'] & B_1 \arrow[r, "B"', "\shortmid"{marking}] & B_2 \arrow[r, equal] & B_2 \end{tikzcd}$$

is a tight isomorphism.

In $\mathbb{Q}t(\mathfrak{K})$, α is a companion commuter iff it is an invertible cell in $\mathfrak{K}$.

Double pseudofunctors from regular indexed structures

Proposition

Frobenius $\Rightarrow$ *each* $\mu^{\bullet}_{X,Y}$ *for* $(\mathcal{C}, L, R)$*-spans* X, Y *is a companion commuter square in* $\mathbb{Q}t(\mathfrak{K})$*.*

Proof.

Idea: $\mu^{\bullet}$ is built out of $\mu^{-1}_{x_1,y_1}$ and the *mate* of μ_{x_2,y_2}, so it is enough to show the latter is invertible as a cell in $\mathfrak{K}$.

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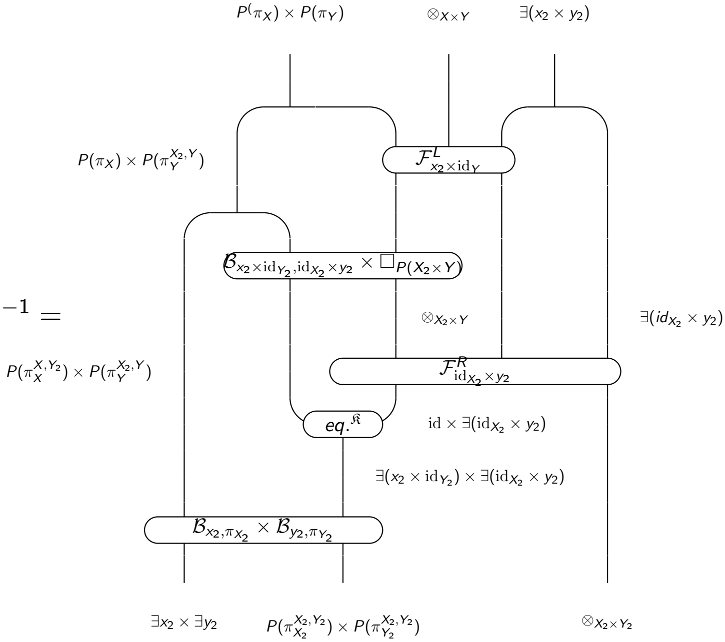
We can build this inverse out of Beck-Chevalley and Frobenius cells. $\square$

$$\mathsf{mate}(\mu_{x_2,y_2})^{-1} =$$



Double pseudofunctors from regular indexed structures

Putting everything together, a $(\mathcal{C}, L, R)$-regular indexed monoidal structure yields a lax symmetric monoidal double pseudofunctor $\mathbb{S}pan(\mathcal{C}, L, R)^{\mathrm{op}} \to \mathbb{Q}t(\mathfrak{K})$ whose laxitors are companion commuters.

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For the converse, we will happily assume the Grandis-Paré's Strictification Theorem (7.5 in [5]) has been applied first, and show how the tight component of a lax symmetric monoidal double functor $\mathbb{S}pan(\mathcal{C}, L, R)^{\mathrm{op}} \to \mathbb{Q}t(\mathfrak{K})$ whose laxitors are companion commuters is indeed a regular monoidal structure.

Indexed structures out of double functors

Fix a lax symmetric monoidal double functor $Q \colon \mathbb{S}pan(\mathcal{C}, L, R)^{\mathrm{op}} \to \mathbb{Q}t(\mathfrak{K})$ whose laxitors are companion commuters.

Step 1. Normal double functors preserve companions and conjoints: as every tight f in $\mathbb{S}pan(\mathcal{C}, L, R)$ has a companion f^* and a conjoint $f_!$, we have $Q(f_!) \vdash Q(f)$ as maps in $\mathfrak{K}$.

Indexed structures out of double functors

Step 2. Note that the Beck-Chevalley property follows from the fact that spans compose by pullback: given a $(\mathcal{C}, L, R)$-pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "k"] \\ X \arrow[r, "g"'] & Y \end{tikzcd},$$

we can write

$$\begin{tikzcd} & & A \arrow[dl, "h"'] \arrow[dr, "f"] \arrow[d, phantom, very near start, "\vee"] & & \\ & X \arrow[dl, equal] \arrow[dr, "g"] & {} & B \arrow[dl, "k"'] \arrow[dr, equal] & \\ X \arrow[rr, "g_!"', "|" marking] & & Y \arrow[rr, "k^*"', "|" marking] & & B \end{tikzcd} \cong h^* \odot f_!,$$

and the double functor maps these to $Q(k)\exists(g) \cong \exists(f)Q(h)$. This iso is in fact the Beck-Chevalley cell.

Indexed structures out of double functors

Step 3. Since Q is lax symmetric monoidal, we can transfer the monoidal structure to the fibres: $\otimes_A \colon QA \times QA \to QA$ is given by $QA \times QA \xrightarrow{\mu_{A,A}} Q(A \times A) \xrightarrow{Q(\Delta_A)} QA$, the monoid units are $I_A := 1 \xrightarrow{I} Q(1) \xrightarrow{Q(!_A)} QA$.

Indexed structures out of double functors

Step 4. Use the inverses of $\mu_{x,y}$ (which exists as the μ cells are companion commuters) to construct the Frobenius cells:

$$
\begin{tikzcd}
Q(B\times A) \arrow[rr, equal] \arrow[d, equal] \arrow[drr, phantom, "\circlearrowleft"] & & Q(B\times A) \arrow[rr, "Q(f^*)\times I_{QA}"] \arrow[d, "\mu_{B,A}"'] \arrow[drr, phantom, "\Uparrow \mu_{f^*,I_A}"] & & QA\times QA \arrow[rr, "\mu_{A,A}"] \arrow[d, "\mu_{A,A}"] \arrow[drr, phantom, "\circlearrowleft"] & & Q(A\times A) \arrow[d, "Q(\Delta_A)"] \\
Q(B\times A) \arrow[rr, "\mu_{B,A}"] \arrow[d, equal] \arrow[drr, phantom, "\circlearrowleft"] & & Q(B\times A) \arrow[rr, "Q(f^*\times I_A)"] \arrow[d, equal] \arrow[drrrr, phantom, "\circlearrowleft\ (\star)"] & & Q(A\times A) \arrow[rr, "Q(\Delta_A^*)"] & & QA \arrow[d, "Q(f_!)"] \\
QB\times QA \arrow[rr, "\mu_{B,A}"] \arrow[d, equal] & & Q(B\times A) \arrow[rr, "Q(I_B\times f_!)"] \arrow[drrr, phantom, "\circlearrowleft"] & & Q(B\times B) \arrow[r, equal] & Q(B\times B) \arrow[r, "Q(\Delta_B^*)"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & QB \arrow[d, equal] \\
QB\times QA \arrow[r, equal] & QB\times QA \arrow[rr, "I_B\times Q(f_!)"'] & & QB\times QB \arrow[rr, "\mu_{B,B}"'] & & Q(B\times B) \arrow[r, "Q(\Delta_B^*)"'] & QB
\end{tikzcd},
$$

where the cell in the bottom-left is the companion of $\mu_{I_B,f_!}$, and the cell $\star$ in the middle is an isocell by Beck-Chevalley

Logical systems

DJM: Since each $P^\bullet$ is a lax symmetric monoidal double pseudofunctor, we can form the following pullback:

$$\begin{tikzcd}
\mathbb{C}ert(P)^{\mathrm{coop}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Q}t(* \downarrow^{colax} \mathcal{C}at) \arrow[d, "\mathbb{Q}t(d_0^*)"] \\
\mathbb{S}pan(\mathcal{C})^{\mathrm{op}} \arrow[r, "P^\bullet"'] & \mathbb{Q}t(\mathcal{C}at)
\end{tikzcd}$$

This defines an interface symmetric monoidal double category for a system.

Food for thought

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5. In a sense, these double functors emerging from hyperdoctrines are trivial: the loose component is fully induced by the tight component. Perhaps there are more interesting examples where the tight and loose components contribute to the logic in equal measure.

Thank you for your time!

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