Categorical differentiation and Goodwillie polynomial functors

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Part I: Abelian functor calculus

This provides an abstract framework that makes certain analogies between classical and functor calculus explicit.

Bauer, Johnson, Osborne, Riehl & Tebbe
Directional Derivatives and Higher Order Chain Rules
Topology and Its Applications, 2018

BJORT

Abelian functor calculus

Let *AbCat* be the category of abelian categories and functors between them. Abelian functor calculus studies functors which are homologically degree n.

- abelian categories: hom sets are abelian groups, every category has 0, ker / coker are well-behaved.
- $F(x) \cong F(0) \oplus cr_1F(x)$, $cr_1F(x)$ detects the failure of F to be reduced.
- $cr_1F(x \oplus y) \cong cr_1F(x) \oplus cr_1F(y) \oplus cr_2F(x,y)$ cr_2F detects the failure of F to be additive.
- cr_nF defined recursively by Eilenberg-MacLane.

In abelian functor calculus, $F : \mathcal{A} \to \mathrm{Ch}\mathcal{B}$ is *degree n* if $\mathrm{cr}_{n+1}F$ is contractible.

The Taylor Tower

There is a Taylor series-like tower of approximations[1]

$$\begin{tikzcd} & & F \arrow[dll] \arrow[dl] \arrow[d] \arrow[drr] & & \\ P_0F & P_1F \arrow[l, "q_1"'] & P_2F \arrow[l, "q_2"'] & \cdots \arrow[l] & P_\infty F \arrow[l] \end{tikzcd}$$

There is an adjoint pair of functors

$$C_n \circlearrowright \mathrm{Fun}(A,B) \underset{cr_n}{\overset{\Delta_*}{\rightleftarrows}} \mathrm{Fun}(A^n, B)$$

and P_nF is the Bar construction associated to C_n

$$\to C_n^3F \to C_n^2F \to C_nF \to F \to 0$$

P_nF is the 'closest' degree n functor to F.

[1] B. Johnson & R. McCarthy, Deriving Calculus with Cotriples, Trans AMS, 2003.

The Taylor Tower

There is a Taylor series-like tower of approximations

$$\begin{tikzcd}
 & & & F \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[dr] & \\
P_0 F & P_1 F \arrow[l, "q_1"'] & P_2 F \arrow[l, "q_2"'] & \cdots \arrow[l] & P_\infty F \arrow[l]
\end{tikzcd}$$

In some cases, this notion of degree n is the same as Goodwillie's notion of n-excisive. [2]

If F commutes with the geometric realization functor,

these are the same.

[2] Bauer, Johnson, McCarthy, Cross effects and calculus in an unbased setting, 2014.

The Taylor Tower

There is a Taylor series-like tower of approximations

$$\begin{tikzcd} & & & F \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[drr] & & \\ P_0F & P_1F \arrow[l, "q_1"'] & P_2F \arrow[l, "q_2"'] & \cdots \arrow[l] & & P_\infty F \arrow[ll] \end{tikzcd}$$

Let $D_nF = \text{fiber}(q_n : P_nF \longrightarrow P_{n-1}F)$. Then D_nF is homogeneous degree n, meaning D_nF is degree n,

$$P_k D_n F \simeq 0 \quad k < n.$$

In particular, D_1F is additive and reduced (linear).

The Directional Derivative

If f is a function of (several) real variables, the directional derivative is

$$\nabla f(v;x) = \lim_{t\to 0}\frac{1}{t}\left(f(x+tv) - f(x)\right).$$

For a functor F of abelian categories, Johnson-McCarthy define

$$\nabla F(V;X) = D_1^V \ker\left(F(V\oplus X)\to F(X)\right).$$

$$\cong D_1^V F(V) \oplus D_1^V cr_2 F(V,X)$$

The Faa di Bruno Formula

If F and G are composable functors, Johnson-McCarthy showed

$$\nabla(FG) \simeq \nabla F(\nabla G, G).$$

chain rule

$\nabla(FG)(V,X) \simeq \nabla F(\nabla G(V,X); GX)$

The nth higher order directional derivative is defined recursively by

$$\Delta_n F(V_1, \ldots, V_n; X) = \nabla(\Delta_{n-1}F)((V_n, \ldots, V_1)(V_{n-1}, \ldots, X))$$

Theorem (BJORT Theorem 8.1)

$$\Delta_n(FG) \simeq \underline{\Delta_n F(\Delta_n G, \ldots, \Delta_1 G; G)}.$$

This is reminiscent of the Faa di Bruno formula for directional derivatives published by Huang, Marcantognini and Young. [3]

Question: why does functor calculus resemble regular calculus?

[3] Chain rules for higher derivatives, Math. Intelligencer, 2006.

Part II: Cartesian Differential Categories

Over the past few centuries, one of the most fundamental concepts in all of mathematics has been differentiation.

R. Blute, R. Cockett, R. Seely
Cartesian Differential Categories
Theory and Applications of Categories 2009

Motivating Example

Let $\mathcal{S}mooth$ be the category with

- Objects: Natural numbers $n \geq 1$ $\mathbb{R}^n$
- Morphisms: smooth maps from $\mathbb{R}^n \to \mathbb{R}^m$

Definition (Differentiation in $\mathcal{S}mooth$)

For $f : \mathbb{R}^n \to \mathbb{R}^m$, the Jacobian Df gives rise to the directional derivative

$$\nabla f(v, a) = Df(a) \cdot v.$$

Differentiation in $\mathcal{S}mooth$

The directional derivative satisfies:

- ∇ is a linear operator
  $\nabla(f+g) = \nabla f + \nabla g$
- $\nabla f(-, a)$ is a linear function
  $\nabla f(v+w, a) = \nabla f(v, a) + \nabla f(w, a)$
- The chain rule
- The Jacobian of $id_{\mathbb{R}^n}$ is $\mathrm{I}_{\mathrm{n}\times\mathrm{n}}$
- The derivative commutes with products in $\mathcal{S}mooth$
  $\langle f, g\rangle : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$
- Derivative of linear functions: $\frac{\partial}{\partial v} f(\vec{x}) \cdot v = f(\vec{v})$
- Mixed partials agree: $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial x \partial y}$

Cartesian differential categories[4]

Definition (Cartesian differential categories)

A Cartesian left additive category with a differential operator
$\nabla : \text{hom}(X, Y) \to \text{hom}(X \times X, Y)$ satisfying:

f $\qquad$ ∇f

- $\nabla(f + g) = \nabla f + \nabla g$
- $\nabla f(v + w, a) = \nabla f(v, a) + \nabla f(w, a)$
- $\nabla(fg)(v, a) = \nabla f(\nabla g(v, a), g(a))$
- $\nabla id(v, a) = v$
- $\nabla(f \times g)(v, a) = \nabla f(v, a) \times \nabla g(v, a)$
- $\nabla(\nabla f)(w, 0, v, a) = \nabla f(w, a)$
- $\nabla(\nabla f)(0, b, v, a) = \nabla(\nabla f)(0, v, b, a)$

[4] Blute, Cockett, Seely, Cartesian Differential Categories, TAC, 2009.

Theorem (BJORT Theorem 6.5)

The (homotopy) category of abelian categories is a Cartesian differential category.

The derivative of a functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories is

$$\nabla F(V, A) = D_1 F(V) \oplus D_1^V cr_2 F(V, A)$$

① ∇ is a linear operator since

$$\nabla(F \oplus G)(V, A) = D_1(F \oplus G)(V) \oplus D_1^V cr_2(F \oplus G)(V, A)$$
$$= D_1 F(V) \oplus D_1 G(V) \oplus D_1^V cr_2 F(V, A) \oplus D_1^V cr_2 G(V)$$

② $\nabla F(-, A)$ is additive + reduced in the V-variable

b/c $= D_1^V (F(V) \oplus cr_2 F(V, A))$

③ The chain rule $\nabla(FG)(V, A) \simeq \nabla F(\nabla G(V, A), G(A))$

Part III: Goodwillie calculus

We make precise the analogy between Goodwillie's calculus of functors in homotopy theory and the differential calculus of smooth manifolds by introducing a higher-categorical framework of which both theories are examples.

Bauer, Burke, Ching
ArXiV:2101.07819v1
2021

n-excisive functors

Let Top be the category of topological spaces. $F : \mathrm{Top} \longrightarrow \mathrm{Top}$

Definition (Excisive)

A functor F is excisive if it takes homotopy pushouts to homotopy pullbacks. In particular, if F is also reduced then it is linear:

$$F(X \vee Y) = F(X) \times F(Y).$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X +_A Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(X) \arrow[d] \\ F(Y) \arrow[r] & F(X +_A Y) \end{tikzcd}$$

$F(A) \simeq F(X) \times_{F(X +_A Y)} F(Y)$

when $A = *$, $X +_A Y = X \vee Y$

Functor Calculus

$F \longrightarrow P_n F$

$\circlearrowleft \mathrm{Fun}(\mathrm{Top},\mathrm{Top}) \underset{\Delta}{\overset{cr_n}{\rightleftarrows}} \mathrm{Fun}(\mathrm{Top}^n,\mathrm{Top})$

There is a Taylor series-like tower of approximations

$P_nF : \mathrm{hocof}\left(|\underbrace{C_n^{*+1}} F| \to F \right)$

$$\begin{tikzcd}
0 \arrow[r] & {|C_n^{*+1} F|} \arrow[r] & F \arrow[r] \arrow[r, bend left] & P_nF \arrow[d] \\
& & & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& & F \arrow[dll] \arrow[dl] \arrow[d] \arrow[drr] & & \\
P_0F & P_1F \arrow[l, "q_1"'] & P_2F \arrow[l, "q_2"'] & \cdots \arrow[l] & P_\infty F \arrow[l]
\end{tikzcd}$$

where P_nF is the best n-excisive approximation to F.

Theorem: $D_nF(X) = \mathrm{hofib}(q_n)$

$\simeq \partial_n F(x) \wedge_{h\Sigma_n} X^{\wedge n}$ $\quad \frac{f^{(n)}(a)\,x^n}{n!}$

$\Sigma_n \circlearrowleft$

why does Goodwillie's calculus behave so much like regular calculus?

Tangent categories[5]

Definition (Tangent Categories)

A tangent category $(\mathcal{X}, T)$ consists of an endofunctor $T : \mathcal{X} \to \mathcal{X}$ together with:

$TM \longrightarrow M$

- the projection $p : T \to Id$
- the zero section $0 : Id \to T$
- the addition $+ : T \times_{Id} T \to T$
- the canonical flip $c : T^2 \to T^2$
- the vertical lift $\ell : T \to T^2$

A large collection of diagrams are required to commute.

These natural transformations make TM into a bundle of commutative monoids over M which resembles the tangent space of a manifold.

[5] Cockett-Cruttwell, Differential Structure, Tangent Structure and SDG, Applied Categorical Structures, 2014

Let $\mathbb{C}at_{\infty}^{diff}$ be the subcategory of the ∞ category of ∞ categories whose morphisms are functors that preserve sequential colimits.[6]

Theorem (B.-Burke-Ching 2022)

$\mathbb{C}at_{\infty}^{diff}$ is a tangent infinity category, and the tangent functor is given by the excisive functors

$$T(\mathcal{C}) := Exc(\mathcal{S}_{fin,*}, \mathcal{C})$$

where $\mathcal{S}_{fin,*}$ is the category of finite simplicial sets.

[6] B., Burke, Ching, Tangent ∞ categories and Goodwillie calculus, https://arxiv.org/abs/2101.07819

THANK YOU!!

References:

- Bauer, Burke, Ching, *Tangent ∞ categories and Goodwillie calculus*, https://arxiv.org/abs/2101.07819
- Bauer, Johnson, Osborne, Riehl, Tebbe, *Directional derivatives and higher order chain rules for abelian functor calculus*, Topology and Its Applications, 2016.
- Blute, Cockett, Seely, *Carteisan differential categories*, Theory and Applications of Categories, 2009.
- Huang, Marcantognini, Young, *Chain rules for higher derivatives*, The Mathematical Intelligencer, 2006.
- Johnson, McCarthy, *Deriving calculus with cotriples*, Trans AMS, 2003.