# The matrix product of coloured symmetric sequences

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## AIM

- to extend 'matrix multiplication' from symmetric sequences to coloured symmetric sequences
- to show that we obtain a monoidal structure on the bicategory of coloured symmetric sequences.

## MOTIVATION

- application to Boardman-Vogt tensor product
- new example of monoidal bicategory

# OUTLINE

I. Species of structures and symmetric sequences

II. Coloured symmetric sequences

III. Proof strategy

# I SPECIES OF STRUCTURES & SYMMETRIC SEQUENCES

$\mathbb{B}$ = category

- objects : finite sets $(U, V, \dots)$
- maps : bijections $(\sigma : U \to V, \dots)$

**Definition (Joyal '82)** A species of structures is a functor

$$F : \mathbb{B} \longrightarrow \underline{\mathrm{Set}}$$

$$\begin{tikzcd}
U \arrow[r, mapsto] \arrow[d, "\sigma"'] & F[U] \arrow[d, "{F[\sigma]}"] \\
V \arrow[r, mapsto] & F[V]
\end{tikzcd}$$

$F[U]$ ⇝ the set of F-structures on $U$

$F[\sigma]$ ⇝ re-labelling

$F[V]$ ⇝ the set of F-structures on $V$

# Analytic functors ( Joyal '86 )

Let $F : \mathbb{B} \longrightarrow \underline{\mathbf{Set}}$. Define the analytic functor

$$\begin{array}{ccc} \underline{\mathbf{Set}} & \longrightarrow & \underline{\mathbf{Set}} \\ X & \longrightarrow & \sum_{n \in \mathbb{N}} F[n] \times X^n /_{\simeq} \end{array}$$

quotient by $S_n$

Idea : functorial counterpart of

$$f(x) = \sum_{n \in \mathbb{N}} f_n \frac{x^n}{n!}$$

# The calculus of species of structures

Joyal ('82, '86) defined several operations

$$F + G \ , \quad F \cdot G \ , \quad G \circ F \ , \quad F'$$

substitution

Maia & Méndez (2008) defined a new operation

$$F \,\square\, G$$

arithmetic product

Note Many laws relating these operations.

# Arithmetic product

Let $F, G : \mathbb{B} \longrightarrow \underline{\mathbf{Set}}$ s.t. $F[\emptyset] = G[\emptyset] = \emptyset$. Define

$$(F \,\square\, G)[U] = \sum_{(\pi,\tau) \in R[U]} F[\pi] \times G[\tau]$$

"rectangles on $U$" (suitable partitions)

**Example** $U = \{a, b, c, d, \dots, \ell\}$

| | E | C | B | G |
|---|---|---|---|---|
| A | a | g | i | k |
| F | c | b | h | j |
| D | f | e | d | ℓ |

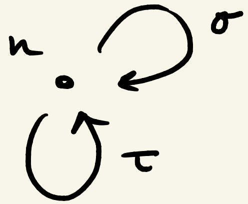
$\pi = \{A, F, D\}$

$\tau = \{E, C, B, D\}$

# Symmetric sequences

- $\mathbb{S}_n$ = n-th symmetric group

  $\Rightarrow$ viewed as a category



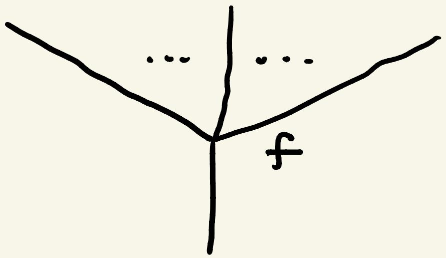

- $\mathbb{S} = \coprod_{n \in \mathbb{N}} \mathbb{S}_n$

**Definition** A symmetric sequence is a functor

$$F : \mathbb{S} \longrightarrow \underline{\mathrm{Set}}$$

**Idea**

$$f \in F[n] \quad \Longleftrightarrow$$



## Remarks

- $\mathbb{B} \simeq \mathbb{S}$
- Species of structures $\simeq$ Symmetric sequences
- Operations on species $\Longleftrightarrow$ operations on symmetric sequences

  Example: Substitution on species corresponds to the monoidal structure on $[\mathbb{S}, \underline{Set}]$ whose monoids are symmetric operads (Kelly '72).

- Combinatorics vs algebraic topology.

## Matrix multiplication of symmetric sequences (Dwyer - Hess)

Let $F, G : \mathbb{S} \longrightarrow \underline{Set}$. Define

$$(F \,\square\, G)[n] = \sum_{l \cdot m = n} F[l] \times G[m] \times_{S_l \times S_m} S_n$$

**Note** Essentially the arithmetic product on species.

**Theorem (Dwyer & Hess)** The matrix product determines a symmetric monoidal structure on $[\mathbb{S}, \underline{Set}]$.

## Interchange

Dwyer & Hess also noted

$$(G_1 \circ F_1) \square (G_2 \circ F_2) \xrightarrow{\sigma} (G_1 \square G_2) \circ (F_1 \square F_2)$$

and made a conjecture.

**Theorem (Garner & Lopez-Franco, 2016)**

The interchange map relates the substitution and the matrix product monoidal structures so as to make $[\mathbb{S}, \underline{\mathrm{Set}}]$ into a duoidal category.

**Note**

$$(B_1 \times A_1) + (B_2 \times A_2) \longrightarrow (B_1 + B_2) \times (A_1 + A_2) .$$

# II. COLOURED SYMMETRIC SEQUENCES

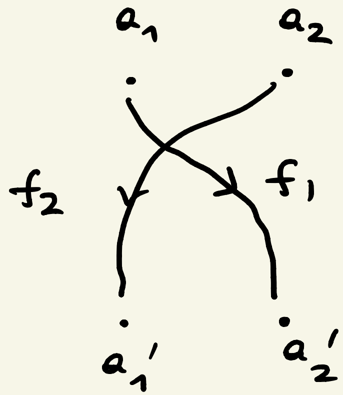
Let $A$ be a small category.

$S_n(A)$ = category with

- <u>objects</u> : $(a_1, \dots, a_n)$, $a_i \in A$
- <u>maps</u>

$$(\sigma, f_1, \dots, f_n) : (a_1, \dots, a_n) \longrightarrow (a_1', \dots, a_n')$$

where $\sigma \in S_n$, $f_i : a_i \longrightarrow a'_{\sigma(i)}$.




$$S(A) = \coprod_{n \in \mathbb{N}} S_n(A)$$

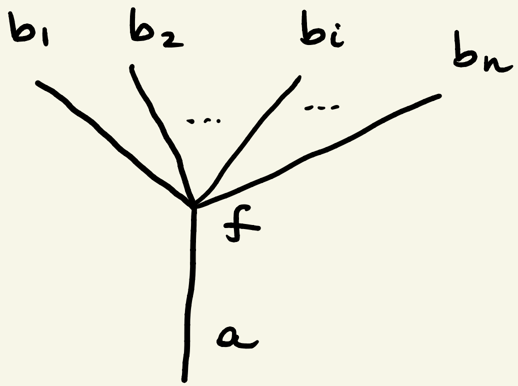
<u>Note</u> $\mathbb{S} = S(\mathbb{1})$.

**Definition** Let $A, B$ be small categories.

- A categorical symmetric sequence $F : A \rightsquigarrow B$ is a functor $F : S(B)^{op} \times A \longrightarrow \underline{\mathrm{Set}}$
- If $A, B$ are sets, we have a coloured symmetric sequence.

**Idea** :

$$f \in F[b_1, \dots, b_n ; a] \quad \Longleftrightarrow$$




Operations are many-sorted.

## Theorem (Fiore, Gambino, Hyland, Winskel 2008)

Small categories and categorical symmetric sequences are the objects and maps of a bicategory.

$$\boxed{A \overset{F}{\rightsquigarrow} B \quad =_{\mathrm{def}} \quad S(B)^{op} \times A \longrightarrow \underline{\mathrm{Set}}}$$

There is a full sub-bicategory of coloured symmetric sequences spanned by sets.

**Note** Monoids in the bicategory of coloured symmetric sequences = coloured sym. operads.

**Goal** : Extend the matrix multiplication of Dwyer & Hess to categorical / coloured symmetric sequences.

Let

$$F : A \rightsquigarrow B \iff F : S(B)^{op} \times A \longrightarrow \underline{Set}$$

$$G : C \rightsquigarrow D \iff G : S(D)^{op} \times C \longrightarrow \underline{Set}$$

**Want** :

$$F \square G : A \times C \rightsquigarrow B \times D \iff$$

$$F \square G : S(B \times D)^{op} \times (A \times C) \longrightarrow \underline{Set}$$

$$\vec{u} \quad , \quad (a, c) \longmapsto \ ?$$

$$(F \,\square\, G)\big(\vec{u}\,;\,(a,b)\big) \;=_{\mathrm{def}}$$

$$\int^{\vec{b} \in S(B),\ \vec{d} \in S(D)} F\big[\overset{(b_1,\dots,b_\ell)}{\vec{b}}\,;\,a\big] \;\times\; G\big[\overset{(d_1,\dots,d_m)}{\vec{d}}\,;\,c\big]$$

$$S(B \times D)\Big[\vec{u}\,,\ \big((b_1,d_1),(b_1,d_2),\dots,(b_1,d_m),\ (b_2,d_1),(b_2,d_2),\dots,(b_2,d_m),\ \dots,\ (b_\ell,d_1),(b_\ell,d_2),\dots,(b_\ell,d_m)\big)\Big].$$

**Note** Generalises 'rectangles' & Dwyer-Hess.

**Theorem** (Gambino, Garner, Vasilakopoulou)

The operation of matrix multiplication gives an oplax monoidal structure on the bicategory of categorical / coloured symmetric sequences.

Coherence conditions.

**Note** Oplax monoidal means that we have

$$(G_1 \circ F_1) \,\square\, (G_2 \circ F_2) \longrightarrow (G_1 \,\square\, G_2) \circ (F_1 \,\square\, F_2)$$

generalising the interchange law seen before.

## III. PROOF STRATEGY

Construct an oplax monoidal structure on a

double category

and then apply results of Garner & Gurski, Shulman, Wester Hansen & Shulman to obtain an oplax monoidal structure on the horizontal bicategory.

Key advantage Less coherence, more 2-categorical.

**Definition** A **double category** $\mathbb{C}$ consists of :

- objects : $A, B, C, \ldots$
- vertical arrows : $f : A \longrightarrow A'$
- horizontal arrows : $M : A \to\!\!\!\!\!+ \; B$ ..
- squares

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ A' \arrow[r, "M'"', "\shortmid"{marking}] & B' \end{tikzcd}$$

\+ compositions , units

\+ axioms

**NOTE** Here, horizontally weak.

**Example** The double category $\underline{\mathbb{P}\mathrm{rof}}$

- <u>objects</u> : small categories

- <u>vertical arrows</u> : functors

- <u>horizontal arrows</u> : profunctors

$$M : A \nrightarrow B \quad =_{\text{def}} \quad M : B^{op} \times A \longrightarrow \underline{\mathrm{Set}}$$

- <u>squares</u> :

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "G"] \\ A' \arrow[r, "M'"', "\shortmid"{marking}] & B' \end{tikzcd} \quad \Longleftrightarrow \quad \alpha_{b,a} : M(b,a) \longrightarrow M'(Gb, Fa) .$$

**Note** Often, we have

$$\begin{tikzcd} A \arrow[d, "f"'] \\ A' \end{tikzcd} \longmapsto A \overset{f_!}{\longrightarrow\!\!\!\!\!\!\!+\;\;} A' \quad , \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & A \arrow[d, "f"] \\ A \arrow[r, "f_!"', "|" marking] & A' \end{tikzcd} \quad \begin{tikzcd} A \arrow[r, "f_!", "|" marking] \arrow[d] \arrow[dr, phantom, "\varepsilon"] & A' \arrow[d, equal] \\ A' \arrow[r, equal] & A' \end{tikzcd}$$

**Example** In **Prof**, $F : A \longrightarrow A'$ gives

$$F_! : A \longrightarrow\!\!\!\!\!\!\!+\;\; A' \quad \text{by} \quad F_!(a', a) =_{\text{def}} A'(a', Fa)$$

**Key idea** Use "strict" vertical structure to induce "weak" horizontal structure.

**Remark** The operation $A \mapsto S(A) = \coprod_{n \in \mathbb{N}} S_n(A)$ gives

- a 2-monad on $\underline{\mathrm{Cat}}$
- a vertical monad on $\underline{\mathbb{P}\mathrm{rof}}$
- a horizontal monad on $\underline{\mathbb{P}\mathrm{rof}}$.

**Fact** Kleisli double category $\mathbb{K}\mathrm{l}(S) = \mathbb{C}\mathrm{at}\,\mathrm{Sym}$

- $\underline{\text{objects}}$: small categories
- $\underline{\text{vertical maps}}$: functors
- $\underline{\text{horizontal maps}}$: categorical symmetric sequences

$$M : A \rightsquigarrow B \;=\; M : A \nrightarrow SB \;=\; M : SB^{op} \times A \longrightarrow \underline{\mathrm{Set}}.$$

Assume $(\mathbb{C}, \otimes, I)$ is monoidal double category.

**Question** : What structure on a horizontal double monad $S : \mathbb{C} \longrightarrow \mathbb{C}$ do we need to get a monoidal structure on $\mathbb{K}\mathrm{l}(S)$ ?

**Try** to define $\boxtimes$ on $\mathbb{K}\mathrm{l}(S)$ as follows:

- <u>Objects</u>: $A, B \longmapsto A \otimes B$

- <u>Vertical maps</u>: $A \xrightarrow{f} A', \quad B \xrightarrow{g} B' \longmapsto$

$$A \otimes A' \xrightarrow{f \otimes f'} B \otimes B'$$

- <u>Horizontal maps</u>: $A \overset{M}{\rightsquigarrow} B, \quad C \overset{N}{\rightsquigarrow} D$ in $\mathbb{K}\mathrm{l}(S)$ $=$

$$A \xrightarrow{M} S(B), \quad C \xrightarrow{N} S(D) \quad \text{in } \mathbb{C} \longmapsto$$

$$A \otimes C \xrightarrow{M \otimes N} S(B) \otimes S(D) \xrightarrow{?} S(B \otimes D)$$

**Theorem** Let $\mathbb{C}$ be monoidal double category,

$S : \mathbb{C} \longrightarrow \mathbb{C}$ monoidal horizontal double monad, ...

Then $|K|(S)$ admits an oplax monoidal structure.

**Fact** (Hyland - Power 2002) $S : \underline{Cat} \longrightarrow \underline{Cat}$ is monoidal:

$$S(B) \times S(D) \xrightarrow{\varphi_{B,D}} S(B \times D)$$

$$(\vec{b}, \vec{d}) \longmapsto \begin{pmatrix} (b_1, d_1), \ldots, (b_1, d_m), \\ (b_2, d_1), \ldots, (b_2, d_m), \\ \vdots \\ (b_\ell, d_1), \ldots, (b_\ell, d_m) \end{pmatrix}$$

This extends to $\mathbb{P}\mathrm{rof}$.

In **Cat Sym** we obtain exactly the formula for matrix multiplication of categorical symmetric sequences as special case.

$$A \times C \xrightarrow[\quad+\quad]{F \times G} S(B) \times S(D) \xrightarrow[\quad+\quad]{(\varphi_{B,D})_!} S(B \otimes D) \qquad \text{in } \underline{\mathbb{P}\mathrm{rof}}$$

$$(F \,\square\, G)(\vec{u}\,;(a,b)) =_{\mathrm{def}}$$

$$\int^{\vec{b} \in S(B),\ \vec{d} \in S(D)} F[\vec{b}\,;a] \times G[\vec{d}\,;c] \times$$

$$S(B \times D)\Big[\vec{u},\ \big((b_1,d_1),(b_1,d_2),\ldots,(b_1,d_m),$$
$$(b_2,d_1),(b_2,d_2),\ldots,(b_2,d_m),$$
$$\vdots$$
$$(b_\ell,d_1),(b_\ell,d_2),\ldots,(b_\ell,d_m)\big)\Big].$$

# References


- Dwyer-Hess, The Boardman-Vogt tensor product of operadic bimodules, 2014
- Garner & López Franco, Commutativity, 2016
- Maia & Méndez, On the arithmetic product of combinatorial species, 2008.