What is a Functor?

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Outline of talk

- Back-story

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- Classical CT
  1. Adjoints and actions
  2. Adjoint lifting theorem
  3. Beck's "crude" monadicity theorem
  4. Left adjoint monads "=" right adjoint comonads

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  1. Monadic over slice
  2. Comonadic over slice
  3. Comonadic over Set

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Adjoints and actions

Let $T : D \to D$ be a monad; let $G : C \to D$ have a left adjoint $F : D \to C$. The following are equivalent:

- Left T-actions $\alpha : TG \to G$.
- Monad maps $\theta : T \to GF$.
- Right T-actions $\beta : FT \to F$.

Special case: $T = UF$, $\theta = \mathrm{id}$. The left T-action $\alpha : TU \to U$ is

$$U\varepsilon : UFU \to U$$

and the right T-action $\beta : FT \to F$ is

$$\varepsilon F : FUF \to F$$

Adjoint lifting theorem

Let $T : D \to D$ be a monad. Category of algebras $U : D^T \to D$.

- Lifts of functors $G : C \to D$ through $U : D^T \to D$,

$$\begin{array}{ccc} & & D^T \\ & \overset{\hat{G}}{\nearrow} & \downarrow U \\ C & \xrightarrow[G]{} & D, \end{array}$$

  are equivalent to T-algebra structures $\alpha : TG \to G$.

- If in addition C has reflexive coequalizers, then $\hat{G}$ has a left adjoint $\hat{F}$ iff G has a left adjoint F.

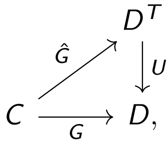
Construction: $\hat{F}(d, \alpha : Td \to d) =$ coequalizer in C:

$$FTd \underset{\beta d}{\overset{F\alpha}{\rightrightarrows}} Fd \longrightarrow \text{“}F \circ_T d\text{”}$$

"Crude" Monadicity Theorem

Theorem: (Beck) A functor $G : C \to D$ is monadic if

- G has a left adjoint F (and then GF is the monad T: canonical left action $G\varepsilon : GFG \to G$);



- C has reflexive coequalizers ($\hat{F} \dashv \hat{G}$), and G preserves them (unit $\hat{\eta} : \mathrm{id} \to \hat{G}\hat{F}$ an isomorphism; note (1));
- G reflects isos (counit $\hat{\varepsilon} : \hat{F}\hat{G} \to \mathrm{id}$ an isomorphism; note (2)).

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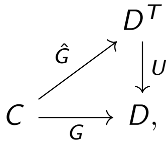

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Crudely monadic functors compose. Even better:

Theorem: If $U : C \to D$ is crudely monadic and $V : D \to E$ is monadic, then $VU : C \to E$ is monadic.

Left adjoint monads = right adjoint comonads

Theorem: (Eilenberg-Moore, 1965)

- If $M : D \to D$ is a monad with a right adjoint $K : D \to D$, then K carries a comonad structure mated to the monad structure on M,

$$\frac{\mu : MM \to M}{\delta : K \to KK}, \qquad \frac{\eta : \mathrm{id} \to M}{\varepsilon : K \to \mathrm{id}}$$

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- M-algebras are equivalent to K-coalgebras,

$$\frac{\alpha : Md \to d}{\gamma : d \to Kd}$$

- If $(F : D \to C) \dashv (U : C \to D) \dashv (G : D \to C)$, then $M = UF \dashv UG = K$ and $\mathrm{Alg}_{UF} \simeq \mathrm{Coalg}_{UG}$.

Example: C-sets

C a category: $(C_0, C_1, d_0 = \mathrm{dom} : C_1 \to C_0, d_1 = \mathrm{cod} : C_1 \to C_0)$.

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- Left adjoint $F_C \dashv U_C$:

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- $U_C F_C X$ in Set/C_0 is the family indexed over $d : C_0$:

$$\left(\sum_{c:C_0} \sum_{f:c \to d} Xc \right)_{d:C_0} = \left(\sum_{f:C_1} X(d_0 f) \right)_{d = d_1 f} = \Sigma_{d_1} d_0^* X$$

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- Monad $U_C F_C = \Sigma_{d_1} d_0^*$ has a right adjoint $K_C = \Pi_{d_0} d_1^*$.

Example: C-sets

Proposition: The functor $U_C : \mathrm{Set}^C \to \mathrm{Set}/C_0$ is crudely monadic:

- Has a left adjoint $F_C : \mathrm{Set}/C_0 \to \mathrm{Set}^C$;
- $U_C : \mathrm{Set}^C \to \mathrm{Set}/C_0$ preserves reflexive coequalizers (colimits in Set^C are computed "pointwise");
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By Eilenberg-Moore (1965), since $M_C = U_C F_C$ has a right adjoint K_C, the functor U_C is also comonadic: $F_C \dashv U_C \dashv G_C$:

$$\begin{pmatrix} X \\ \downarrow \\ C_0 \end{pmatrix} \xmapsto{G_C} \prod_{d:C_0} Xd^{C(-,d)}$$

C-sets: crude comonadicity and Polyfun

Proposition: $U_C : \mathsf{Set}^C \to \mathsf{Set}/C_0$ is crudely comonadic.

Proof: Know U_C is comonadic (right adjoint G_C, reflects isos). U_C also preserves coreflexive equalizers, e.g., by $F_C \dashv U_C$.

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Proposition: $\Sigma_{C_0} : \mathrm{Set}/C_0 \to \mathrm{Set}$ is comonadic, in fact crudely so.

Proof: The comonad is $C_0 \times - : \mathrm{Set} \to \mathrm{Set}$. Function $X \to C_0 =$ coalgebra $X \to C_0 \times X$. Also Σ_{C_0} preserves connected limits, in particular coreflexive equalizers.

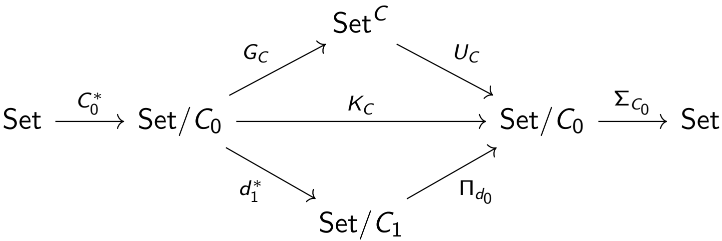
Corollary: The composite

$$\mathrm{Set}^C \xrightarrow{U_C} \mathrm{Set}/C_0 \xrightarrow{\Sigma_{C_0}} \mathrm{Set}$$

is (crudely) comonadic.

C-sets as coalgebras

From $U_C \dashv G_C$ and $\Sigma_{C_0} \dashv C_0^*$, we have $\Sigma_{C_0} U_C \dashv G_C C_0^*$, hence a comonad $p = \Sigma_{C_0} U_C G_C C_0^*$:



so that the polynomial functor

$$p = \left(\text{Set} \xrightarrow{C_1^*} \text{Set}/C_1 \xrightarrow{\Pi_{d_0}} \text{Set}/C_0 \xrightarrow{\Sigma_{C_0}} \text{Set} \right)$$

is a comonad, and

C-sets as coalgebras

Corollary: The category of p-coalgebras is Set^C, with comonadic functor

$$\mathsf{Set}^C \xrightarrow{\quad U_C \quad} \mathsf{Set}/C_0 \xrightarrow{\quad \Sigma_{C_0} \quad} \mathsf{Set}$$

$$(F : C \to \mathsf{Set}) \longmapsto \sum_{c:C_0} Fc$$

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Calculate $p(S)$:

$$\mathsf{Set} \xrightarrow{C_0^*} \mathsf{Set}/C_0 \xrightarrow{K_C} \mathsf{Set}/C_0 \xrightarrow{\Sigma_{C_0}} \mathsf{Set}$$

$$S \longmapsto (C_0^* S)_d = (S)_d \longmapsto \sum_{c:C_0} \prod_{d:C_0} S^{C(c,d)}$$

C-sets as coalgebras

We have

$$\sum_{c:C_0} \prod_{d:C_0} S^{C(c,d)} \xrightarrow{\sim} \sum_{c:C_0} S^{\sum_{d:C_0} C(c,d)} \xrightarrow{\sim} \sum_{c:C_0} S^{d_0^*(c)}$$

so that p-coalgebra structures take the form

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Identify S with a category $\mathrm{El}(F)$ of $F : C \to \mathrm{Set}$: the composite

$$S \to \sum_{c:C_0} S^{d_0^*(c)} \to \sum_{c:C_0} 1 = C_0$$

gives a fibering $S \to C_0$, and for each $f : c \to d$, the corresponding map $S_f : S_c \to S_d$ of fibers takes $s \in S_c$ to $\gamma(s)(f) \in S_d$.

What are functors?

Notation: Polynomial comonad $p : \mathrm{Set} \to \mathrm{Set}$ for C, $q : \mathrm{Set} \to \mathrm{Set}$ for D. Sets $p1, q1$ for C_0, D_0. Left adjoint monad M_p on Set/C_0, right adjoint comonad K_p. Coalgebra categories $\mathrm{Set}_p = \mathrm{Set}^C$.

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- A function $f : p1 \to q1$ together with a lift $F : \mathrm{Set}^C \to \mathrm{Set}^D$ over $\Sigma_f : \mathrm{Set}/p1 \to \mathrm{Set}/q1$:

$$\begin{array}{ccc} \mathrm{Set}_p & \xrightarrow{F} & \mathrm{Set}_q \\ \downarrow{\scriptstyle U_p} & & \downarrow{\scriptstyle U_q} \\ \mathrm{Set}/p1 & \xrightarrow{\Sigma_f} & \mathrm{Set}/q1 \end{array}$$

(same as an M_q-algebra structure on $\Sigma_f U_p$; same as a K_q-coalgebra structure on $\Sigma_f U_p$).

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  (same as an M_q-algebra structure on $\Sigma_f U_p$; same as a K_q-coalgebra structure on $\Sigma_f U_p$).
- This is the concept of *cofunctor* from C to D.

What are functors?

A functor is ...

- (Viewing categories as monads in Span) A function $f : C_0 \to D_0$ together with a 2-cell in Span:

$$\begin{array}{ccc} C_0 & \rightsquigarrow & C_0 \\ {\scriptstyle f}\downarrow & \swArrow_{\psi} & \downarrow{\scriptstyle f} \\ D_0 & \rightsquigarrow & D_0 \end{array} \qquad \begin{array}{ccc} \mathrm{Set}/C_0 & \xrightarrow{M_p} & \mathrm{Set}/C_0 \\ {\scriptstyle \Sigma_f}\downarrow & \swArrow_{\psi} & \downarrow{\scriptstyle \Sigma_f} \\ \mathrm{Set}/D_0 & \xrightarrow[M_q]{} & \mathrm{Set}/D_0 \end{array}$$

  compatible with monad structures, e.g.,

$$\begin{array}{ccccc} \Sigma_f M_p M_p & \xrightarrow{\psi M_p} & M_q \Sigma_f M_p & \xrightarrow{M_q \psi} & M_q M_q \Sigma_f \\ {\scriptstyle \Sigma_f \mu_p}\downarrow & & & & \downarrow{\scriptstyle \mu_q \Sigma_f} \\ \Sigma_f M_p & & \xrightarrow[\theta]{} & & M_q \Sigma_f \end{array}$$

What are functors? Various answers

A functor is ...

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- A pair $(f : p1 \to q1,\ \gamma : f^* U_q \to K_p f^* U_q$ satisfying... K_p-coalgebra conditions on $f^* U_q$!

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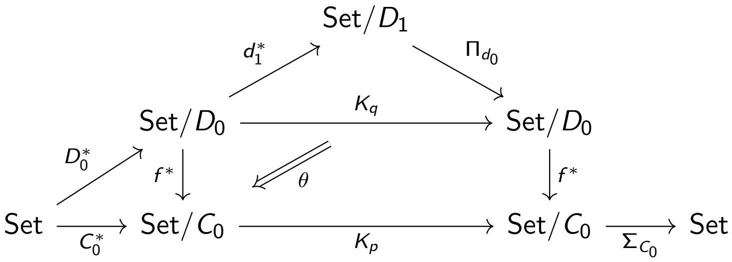
Interpretation: K_p-coalgebra map gives a lift $\mathrm{Set}^F : \mathrm{Set}^D \to \mathrm{Set}^C$:

$$\begin{array}{ccc}
\mathrm{Set}_q & \xrightarrow{\mathrm{Set}^F} & \mathrm{Set}_p \\
{\scriptstyle U_q}\big\downarrow & & \big\downarrow{\scriptstyle U_p} \\
\mathrm{Set}/D_0 & \xrightarrow[f^*]{} & \mathrm{Set}/C_0
\end{array}$$

(Set^F is a left/right adjoint, since f^* is)

What are functors? Enter Beck-Chevalley



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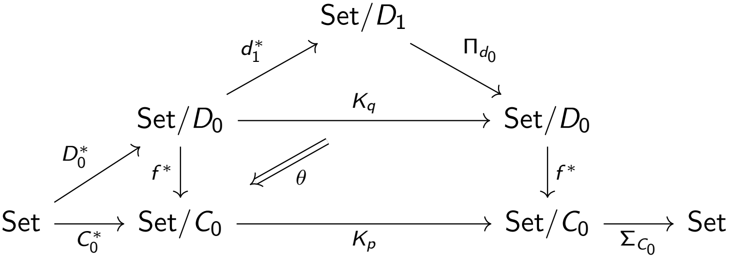
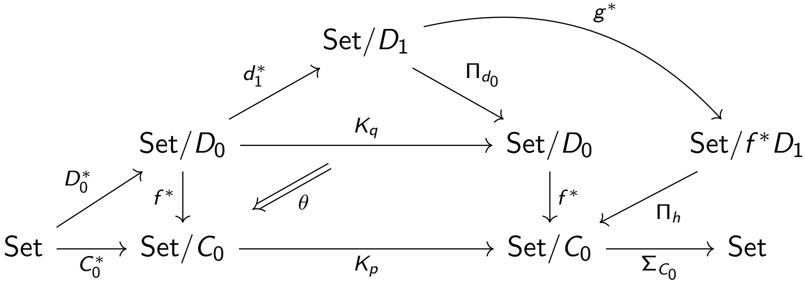
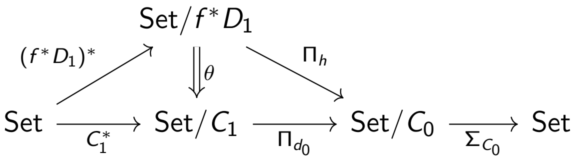
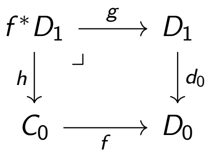




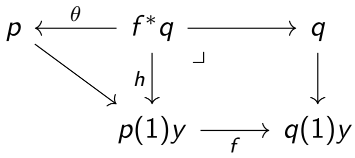
What are functors?

Condensing the previous diagram, let us write





The "top" Polyfun is based on a bundle h obtained as a pullback



Statement: A functor from p to q consists of $f : p(1) \to q(1)$ together with a p-coalgebra structure on f^*q and a p-coalgebra map θ making the triangle commute. "Lens-like."

Thank you!

Notes

Note 1: To prove $\eta : \mathrm{id} \to \hat{G}\hat{F} : D^T \to D^T$ is an iso, it suffices to prove

$$U\eta : U \to U\hat{G}\hat{F} = G\hat{F}$$

is an iso, since $U : D^T \to D$ reflects isos. For an object $(d, \alpha : Td \to d)$ of D^T, we have

$$FTd \underset{\beta d}{\overset{F\alpha}{\rightrightarrows}} Fd \xrightarrow{\text{coeq}} \hat{F}(d, \alpha).$$

Since G is assumed to preserve reflexive coequalizers,

$$GFTd \underset{G\beta d}{\overset{GF\alpha}{\rightrightarrows}} GFd \xrightarrow{\text{coeq}} G\hat{F}(d, \alpha).$$

But this coincides with

$$TTd \underset{\mu d}{\overset{T\alpha}{\rightrightarrows}} Td \xrightarrow{\alpha = \text{coeq}} d.$$

Notes

Note 2: The counit $\hat{F}\hat{G} \to \mathrm{id}$ is given by

$$\begin{array}{ccccc} FTGc & \underset{\varepsilon FGc}{\overset{FG\varepsilon c}{\rightrightarrows}} & FGc & \xrightarrow{\text{coeq}} & \hat{F}\hat{G}c \\ & & & \searrow^{\varepsilon c} & \downarrow \\ & & & & c \end{array}$$

Since G is assumed to reflect isos, it suffices that $G\hat{F}\hat{G} \to G$ be an iso. Since G preserves reflexive coequalizers, the top sequence of

$$\begin{array}{ccccc} GFTGc & \underset{G\varepsilon FGc}{\overset{GFG\varepsilon c}{\rightrightarrows}} & GFGc & \xrightarrow{G(\text{coeq})} & G\hat{F}\hat{G}c \\ & & & \searrow^{G\varepsilon c} & \downarrow \\ & & & & Gc \end{array}$$

is a coequalizer, but then again, as is well-known, the lower sequence is a split coequalizer (QED):

$$GFTGc \underset{\mu Gc}{\overset{GFG\varepsilon c}{\rightrightarrows}} GFGc \xrightarrow{G\varepsilon c} Gc.$$