Sketching the Elephant: 7 Perspectives on Poly

Reed Mullanix
Joint work with Harrison Grodin

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1 Introduction

The category of polynomial functors is full of incredibly rich structure: At time of writing, the author is aware of 5 separate monoidal structures, 2 monoidal closures, 1 right-coclosure, and 2 indexed left-coclosures; see [2] for a full account. This is part of what makes **Poly** appealing to such a disparate group of people: if your problem is vaguely bidirectional and suitably discrete, chances are that **Poly** contains enough structure to model it.

Unfortunately, this makes **Poly** very difficult to study from a structural perspective: there is so much *stuff*[1] that it is difficult to tell if something is fundamental, incidental, or even accidental. A canonical example of this phenomena are the two "duplication" maps $P \to P \triangleleft P$ and $P \to P \otimes P$: are these important to the structural calculus of **Poly**, or should they be ignored entirely?

Such questions are irrelevant when working inside **Poly**, but these sorts of questions are necessary if we want to pin down exactly what a category that "looks like **Poly**" is. If we are able to give a good set of axioms for a **Poly**-like category, then this opens up multiple fronts of research. First, if we work in a generic **Poly**-like category, then we can obviously interpret any constructions or theorems into any suitably structured category: this makes finding new **Poly**-like categories an interesting endeavour. Morover, we can look at the constructions we can perform on **Poly**-like categories: one would expect that they are closed under products, slicing, etc. On the other side, pinning down such a class of categories lets us study the *initial* **Poly**-like category, which lays the groundwork for a hypothetical type theory for polynomial functors. Finally, we can consider modifications of the axioms of **Poly**-like categories: this feels like a good route for understanding how to make **Poly** less discrete while still retaining its richness.

Clearly, pinning down such a class of categories is a useful task. As noted earlier, looking at **Poly** itself is not helpful; it is simply too rich of a theory. Instead, we propose looking at how **Poly** is constructed: at the very least, this ought to give us some idea of where all the structure comes from, and thus how much we ought to privilege it. Somewhat unsurprisingly, we also have many choices of how to construct **Poly**, and each lead to a slightly different perspective on what we ought to consider important. In particular, we have identified 7 different possible constructions:

[1] In the technical sense.

1. **Poly** is the full subcategory of $[\mathbf{Set}, \mathbf{Set}]$ spanned by coproducts of representables.
2. **Poly** is the coproduct completion of the of the product completion of the terminal category; in other words, it is $\Sigma\,\Pi\,1$.
3. **Poly** is the full subcategory of $[\mathbf{Set}, \mathbf{Set}]$ spanned by connected-limit preserving functors.
4. **Poly** is the total category of the fibration $\mathbf{Fam}(\mathbf{Set}^{op})$.
5. **Poly** the category of dependent lenses, IE: pairs of functions $f : B \to B'$ and $f^{\#} : (b : B) \to E'(f(b)) \to E(b)$.
6. **Poly** is the category of single-variable generalized polynomials à la [1], specialized to **Set**.
7. **Poly** is the category of Grothendieck lenses on the family fibration $\mathbf{Fam}(\mathbf{Set})$.

Clearly, all of these definitions give **Poly**, but some privilege different structure, and some generalize in different directions. However, we can group these constructions into 3 larger buckets:

1. Colimit completions.
2. Colimit completions of limit completions.
3. Constructions that use existing structure.

Let us now study each of these groups in further detail.

2 Colimit completions

Constructions 1, 2, 3, 4, and 7 all are some sort of coproduct completion: in fact, construction 1 is simply an explicit description of 4, constructions 2 and 3 are colimit completion over various limit completions, and 7 allows us replaces $\mathbf{Fam}(()\mathbf{Set})$ with an arbitrary Grothendieck fibration E. These constructions all view Σ as the most important part of **Poly**: constructions 1 and 2 additionally privilege set-indexed Σ. Viewing **Poly** in this light yields some interesting insights: to start, it hints as to why **Poly** ends up being discrete. Note that we can consider polynomials with $\mathcal{C}$ valued positions by looking at $\mathbf{Fam}(\mathcal{C}^{op})$. If $\mathcal{C}$ has set-indexed coproducts and a terminal object, then we can construct an $\mathcal{C}$-relative version of $\triangleleft$, but this is only well behaved with respect to y if every object $x : \mathcal{C}$ can be written as a set-indexed coproduct of 1 over the global elements of x. This suggests that trying to build $\triangleleft$ directly after a single colimit completion is going to thwart any attempts to make less discrete version of **Poly**.

3 Colimit completions of limit completions

Constructions 2 and 3 are not only a coproduct completion: it is also a coproduct completion *of a limit completion*. In other words: we start by adding Π over something extremely discrete, and then follow that up by adding Σ. Note that we can also view this as 2 rounds of coproduct completion, as the product completion $\Pi\,\mathcal{C}$ of a category $\mathcal{C}$ is identical to $(\Sigma\,\mathcal{C})^{op}$, the opposite of the coproduct completion. This gives us a hint as to where the composition operator $\triangleleft$ comes from: coproduct completion preserves monoidal structure, and 1 is equipped with a unique (cartesian) monoidal structure. One round of completion turn this cartesian structure on 1 into the cartesian monoidal structure of sets[2], and the second round transforms the product of sets into the monoidal monoidal structure $(\otimes, y)$ on **Poly**. Amazingly, performing 2 rounds of completion *also* creates a new monoidal structure on **Poly**: this is the origin of the composition structure $(\triangleleft, y)$. Moreover, it is known that this doubly iterated process of $\mathbf{Fam}((\mathbf{Fam}(\mathcal{C}))^{op})$ will duplicate every monoidal structure in $\mathcal{C}$, and I conjecture this will always give a normal duoidal structure. As such, $\Sigma\,\Pi\,1$ seems to suggest that normal duoidality is key aspect of **Poly**, as plugging in more complicated categories $\mathcal{C}$ will result in *multiple* normal duoidal structures!

Construction 3 is not immediately obvious as a colimit/limit completion, but this can be seen by how it generalizes. We can replace the category of connected-limit preserving functors with the category of *finite* connected limit preserving functors, which is best viewed as $\mathbf{Fam}((\mathbf{Pro}(\mathbf{Set}))^{op})$. I do not know much about how this interacts with duodality: further exploration is required.

4 Constructions that use existing structure

Constructions 5 and 6 do not perform any completion operations, and instead opts to view **Poly** as something that is occuring in a setting with already existing structure. This seems to suggest that **Poly** is something that is best performed in an ambient dependent type theory, but in my opinion this is somewhat misguided. Dependent type theories should *not* be viewed as LCCCs: instead, LCCCs are conservative over the things we can write in dependent type theories. This is due to the fact that LCCCs give us quantification over all substitutions, not just weakenings. This leads to odd situations like quantifying over contraction, exchange, etc, which do not *really* make sense from a syntactic perspective.[3] As such, this perspective does not seem to clarify exactly what is special about **Poly**: it tells us what structure we need to build it internally, not what structure it ought to have.

References

[1] Gambino, N., and Kock, J. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society 154*, 1 (Jan. 2013), 153–192.

[2] Spivak, D. I. A reference for categorical structures on **Poly**.

[2]Take this with a grain of salt, this document was hastily written!

[3]Perhaps it is useful to study "displayed polynomials" where we restrict ourselves to adjoints to display maps, but this is somewhat orthogonal to the point at hand.