From comonads to calculus

24.06.2021

Topos Colloquium

Joint with Brenda Johnson.

Plan

- What is a "calculus"?

Plan

- What is a "calculus"?
- From comonads to calculi

Plan

- What is a "calculus"?
- From comonads to calculi
- From $\mathcal{P}(n)$-modules to comonads

Plan

- What is a "calculus"?
- From comonads to calculi
- From $\mathcal{P}(n)$-modules to comonads
- The calculus machine for functor categories

What is a calculus?

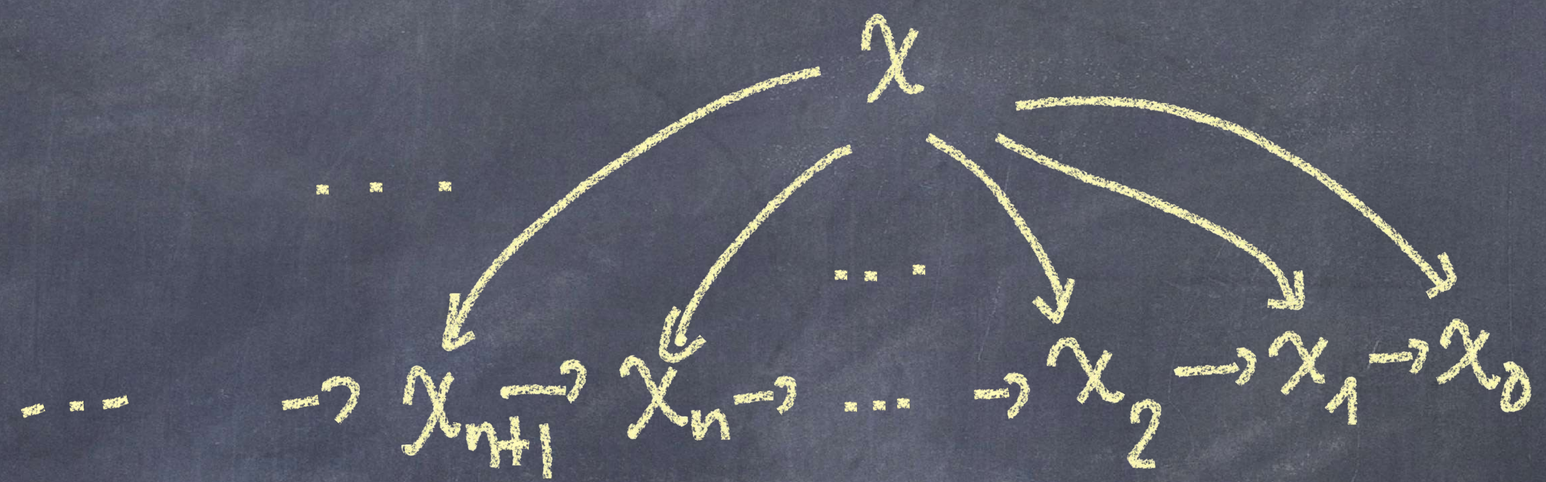
Informal def^n : Given $(\mathcal{M}, WE)$:

$x \in \mathrm{Ob}\,\mathcal{M} \Rightarrow$

$$\cdots \to x_{n+1} \to x_n \to \cdots \to x_2 \to x_1 \to x_0$$

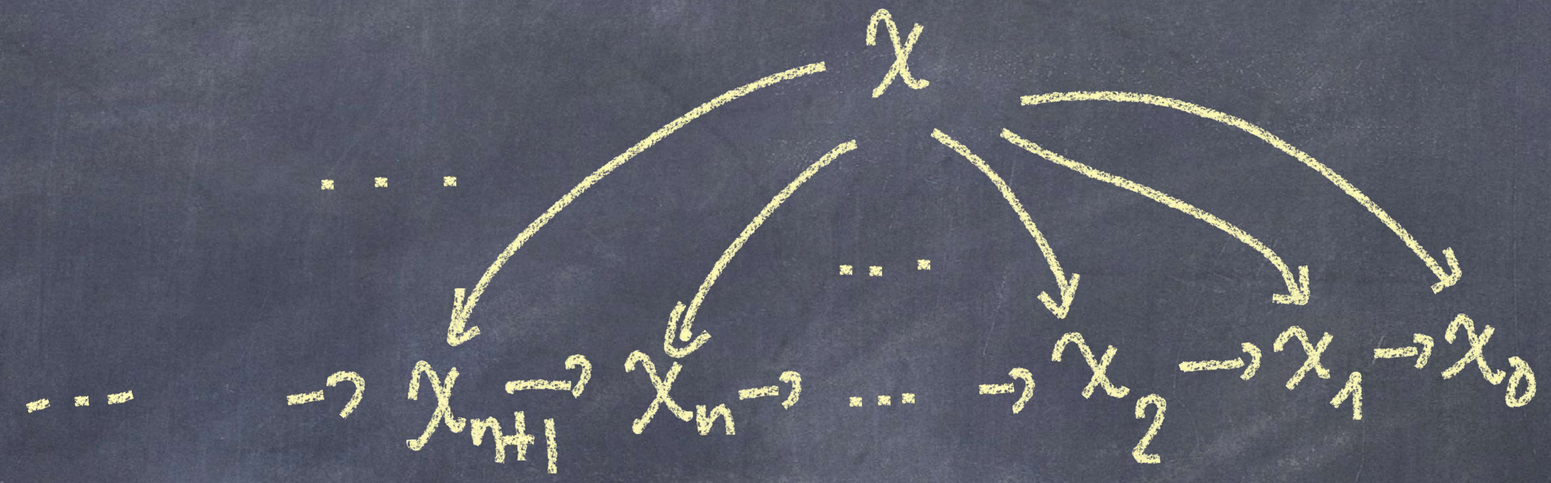
Informal def^n : Given $(\mathcal{M}, WE)$:

$X \in \mathrm{Ob}\,\mathcal{M} \Rightarrow$



Informal def^n : Given $(\mathcal{M}, WE)$:

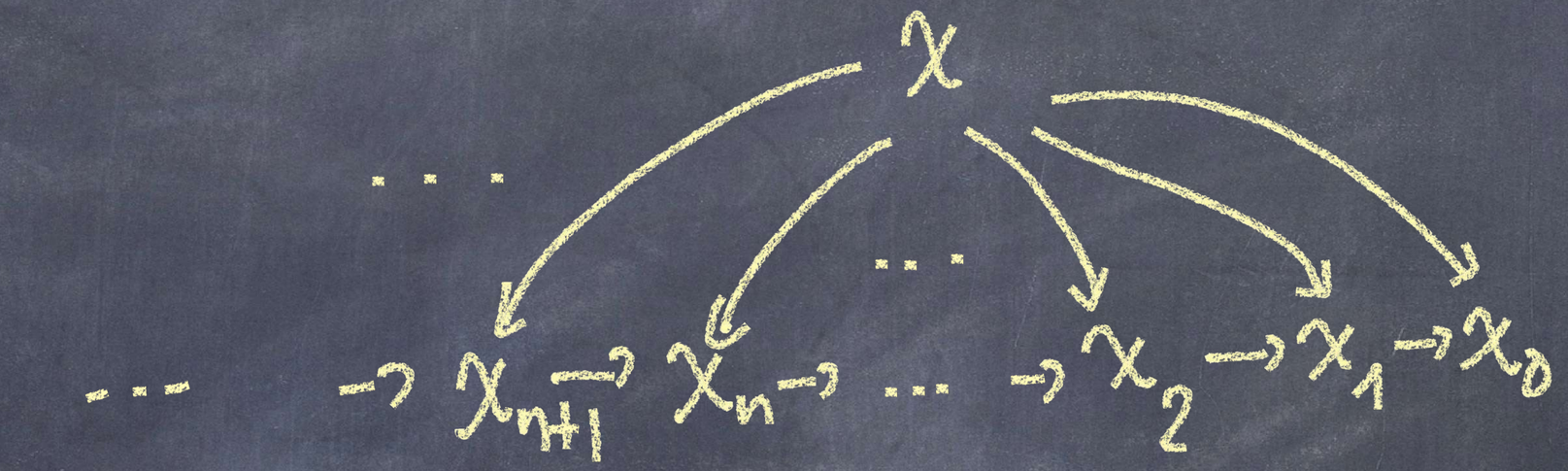
$x \in Ob\,\mathcal{M} \Rightarrow$



such that $x \to x_n$ increasingly close to being a weak equivalence as $n \to \infty$.

Informal defn : Given $(\mathcal{M}, WE)$:

$x \in \mathrm{Ob}\,\mathcal{M} \Rightarrow$



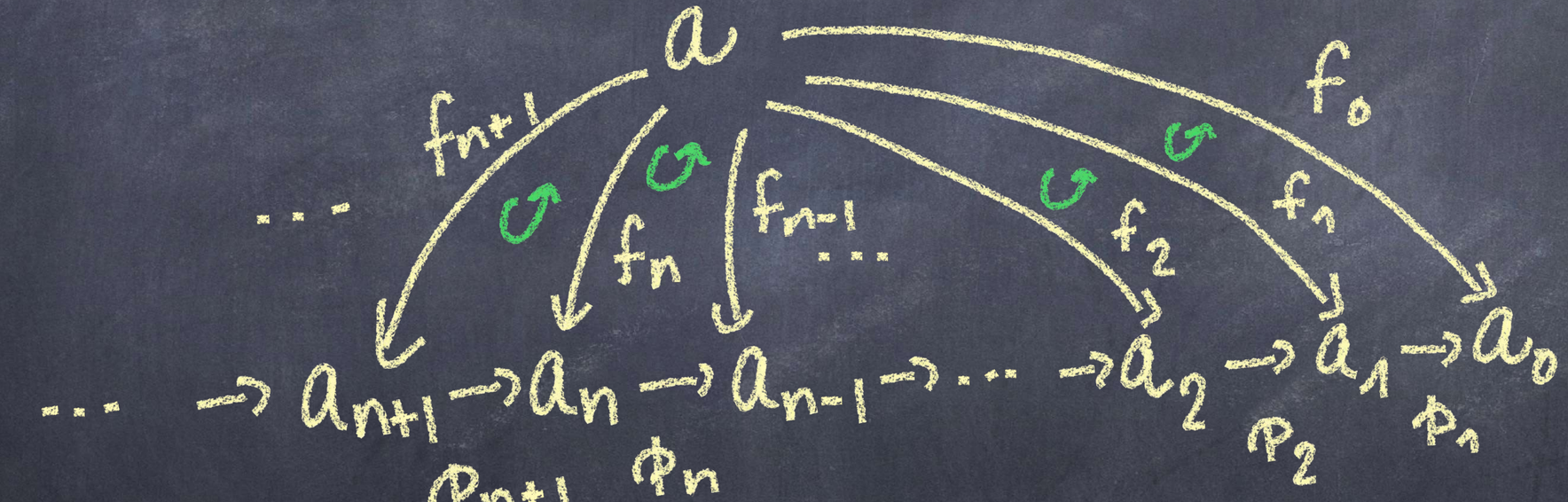
"degree n approx."

such that $x \longrightarrow x_n$ increasingly close to being a weak equivalence as $n \longrightarrow \infty$.

Coaugmented towers

$\mathcal{A}$ - category

$\Rightarrow$ $\mathrm{Tow}^{\vee}(\mathcal{A})$, with objects



Tower functors

$\mathcal{B}$ subcategory of $\mathcal{A}$

$$\Gamma : \mathcal{B} \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{A})$$

$$b \overset{\Gamma}{\longmapsto}$$

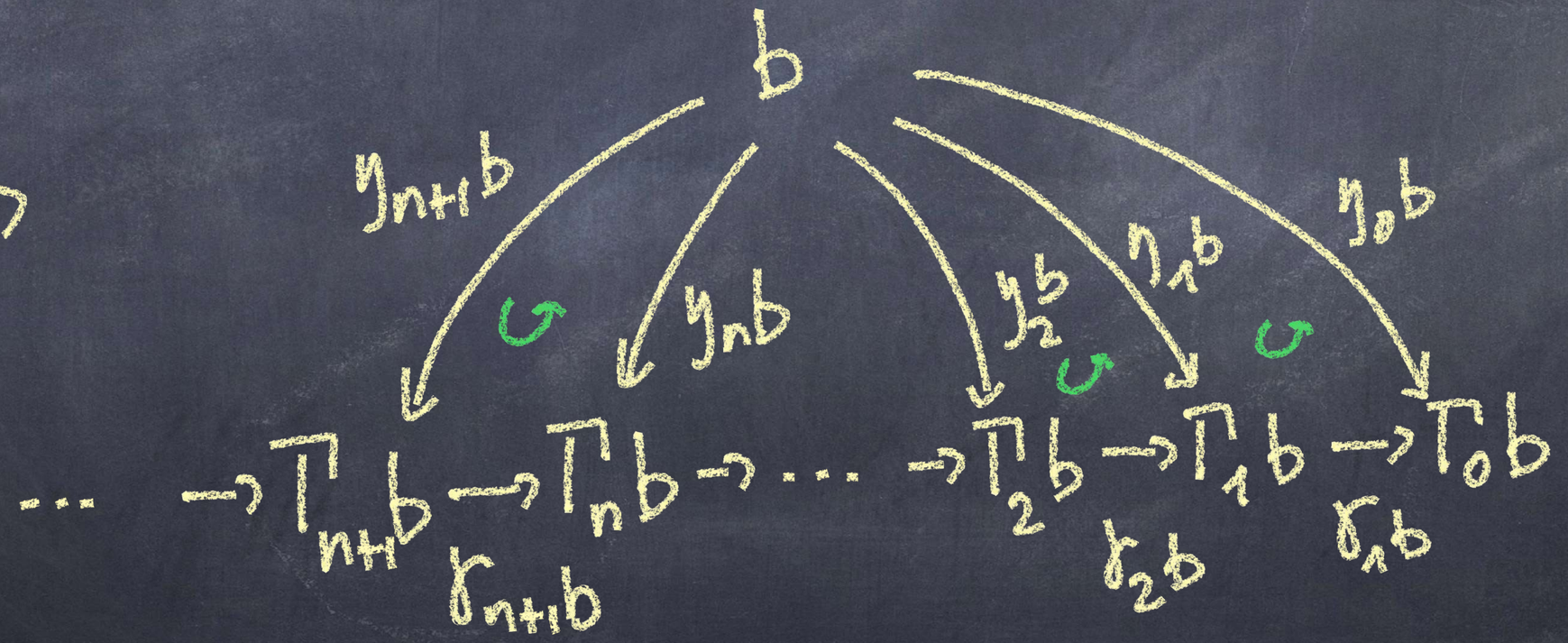
$$\cdots \to \Gamma_{n+1}b \xrightarrow[\gamma_{n+1}b]{} \Gamma_n b \to \cdots \to \Gamma_2 b \xrightarrow[\gamma_2 b]{} \Gamma_1 b \xrightarrow[\gamma_1 b]{} \Gamma_0 b$$

Tower functors

$\mathcal{B}$ subcategory of $\mathcal{A}$

$$\Gamma : \mathcal{B} \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{A})$$

$$b \overset{\Gamma}{\longmapsto}$$



Back to calculus

$(\mathcal{M}, WE)$, $\mathcal{M}' \subseteq \mathcal{M}$

A calculus on $\mathcal{M}'$ with values in $\mathcal{M}$

= tower functor $T : \mathcal{M}' \to \mathrm{Tow}^{\mathcal{D}}(\mathcal{M})$

such that :

Back to calculus

$(\mathcal{M}, WE)$, $\mathcal{M}' \subseteq \mathcal{M}$

A calculus on $\mathcal{M}'$ with values in $\mathcal{M}$

= tower functor $T : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$

such that :

1) $m \leq n \Rightarrow T_m(y_n x) : T_m x \xrightarrow{\sim} T_m T_n x$

(Recall : $y_n x : x \longrightarrow T_n x$)

Back to calculus

$(\mathcal{M}, WE)$, $\mathcal{M}' \subseteq \mathcal{M}$

A calculus on $\mathcal{M}'$ with values in $\mathcal{M}$

= tower functor $T : \mathcal{M}' \to \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

such that:

1) $m \leq n \Rightarrow T_m(y_n x) : T_m x \xrightarrow{\sim} T_m T_n x$

"The degree m approx is equivalent to the degree m approx of the degree n approx. if $m \leq n$."

Back to calculus

$(\mathcal{M}, WE)$, $\mathcal{M}' \subseteq \mathcal{M}$

A calculus on $\mathcal{M}'$ with values in $\mathcal{M}$

= tower functor $T : \mathcal{M}' \to \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

such that:

2) $m \geq n \Rightarrow y_m T_n x : T_n x \xrightarrow{\sim} T_m T_n x$

Back to calculus

$(\mathcal{M}, WE)$, $\mathcal{M}' \subseteq \mathcal{M}$

A calculus on $\mathcal{M}'$ with values in $\mathcal{M}$

= tower functor $T : \mathcal{M}' \to \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$

such that:

2) $m \geq n \Rightarrow y_m T_n x : T_n x \xrightarrow{\sim} T_m T_n x$

"The degree n approx is equivalent to its degree m approx if $m \geq n$."

Back to calculus

$(\mathcal{M}, WE)$, $\mathcal{M}' \subseteq \mathcal{M}$

A calculus on $\mathcal{M}'$ with values in $\mathcal{M}$

= tower functor $T : \mathcal{M}' \to \mathrm{Tow}^{\mathcal{I}}(\mathcal{M})$

such that:

3) $f : x \to x' \in WE \Rightarrow T_n f : T_n x \to T_n x' \in WE$

$\forall n \geq 0.$

Convergence

Let $T : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$ be a calculus.

Let $x \in \mathrm{Ob}\,\mathcal{M}'$.

The tower $T(x)$ converges to x if

$$x \xrightarrow{\sim} \operatorname{holim}_n T_n x .$$

Crucial special case $(\mathcal{M}, WE_{\mathcal{M}})$, $(\mathcal{C}, WE_{\mathcal{C}})$

$\mathrm{fun}(\mathcal{C}, \mathcal{M}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{M})$

Crucial special case $(\mathcal{M}, WE_{\mathcal{M}})$, $(\mathcal{C}, WE_{\mathcal{C}})$

$$fun(\mathcal{C},\mathcal{M}) \subseteq Fun(\mathcal{C},\mathcal{M})$$

A calculus $T: fun(\mathcal{C},\mathcal{M}) \to Tow(Fun(\mathcal{C},\mathcal{M}))$ is a homotopy calculus if

$$\left.\begin{array}{l} F \in fun(\mathcal{C},\mathcal{M}) \\ F(WE_{\mathcal{C}}) \subseteq WE_{\mathcal{M}} \end{array}\right\} \Rightarrow T_n F(WE_{\mathcal{C}}) \subseteq WE_{\mathcal{M}} \quad \forall n \geq 0.$$

Today's talk :

Description of a functorial pipeline for creating (homotopy) calculi on functor categories.

Properties of calculi

$\Gamma : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

- calculus

1) $\forall\ \nu : \mathrm{Id}_{\mathcal{M}} \xrightarrow{\sim} \mathcal{R}$, $\exists$ calculus

$\nu * \Gamma : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

$x \longmapsto$

$$\cdots \longrightarrow \mathcal{R}(\Gamma_n x) \underset{\mathcal{R}(\delta_n x)}{\longrightarrow} \mathcal{R}(\Gamma_{n-1} x) \longrightarrow \cdots$$

Properties of calculi

$\Gamma : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

- calculus

1) $\forall\ \nu : \mathrm{Id}_{\mathcal{M}} \xrightarrow{\sim} \mathcal{R}$, $\exists$ calculus

$\nu * \Gamma : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

$$x \longmapsto \cdots \longrightarrow \mathcal{R}(\Gamma_n x) \xrightarrow[\mathcal{R}(\sigma_n x)]{} \mathcal{R}(\Gamma_{n-1} x) \longrightarrow \cdots$$

$$\nu_{\Gamma_n x} \circ y_n x \qquad x \qquad \nu_{\Gamma_{n-1} x} \circ y_{n-1} x$$

σ

Properties of calculi

$T : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

- calculus

$X \in \mathrm{Ob}\,\mathcal{M}'$ is of degree $\leq n$ if $y_n X : X \xrightarrow{\sim} \Gamma_n X$.

Properties of calculi

$T : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$

- calculus

$X \in \mathrm{Ob}\,\mathcal{M}'$ is of degree $\leq n$ if $y_n X : X \xrightarrow{\sim} \Gamma_n X$

2) $\left.\begin{array}{l} m \geq n \\ y_n X \in WE \end{array}\right\} \Rightarrow y_m X \in WE$.

Properties of calculi

$T : \mathcal{M}' \to \mathrm{Tow}^{\mathcal{Y}}(\mathcal{M})$ - calculus

$X \in \mathrm{Ob}\,\mathcal{M}'$ is of degree $\leq n$ if $y_n X : X \xrightarrow{\sim} \Gamma_n X$.

2) $\left.\begin{array}{l} m \geq n \\ y_n X \in WE \end{array}\right\} \Rightarrow y_m X \in WE$.

"If $m \geq n$ and X is of degree $\leq n$, then X is of degree $\leq m$."

Properties of calculi

$\mathbb{T} : \mathcal{M}' \longrightarrow \mathbb{T}\mathrm{ow}^{\mathcal{G}}(\mathcal{M})$

- calculus

($\mathcal{M}$ pointed)

$X \in \mathrm{Ob}\,\mathcal{M}'$ is **homogeneous of degree n** wrt $\mathbb{T}$ if:

- X of degree $\leq n$

and

- $\mathbb{T}_{n-1} X \cong *$.

Properties of calculi

$T : \mathcal{M}' \longrightarrow Tow^{\mathbb{N}}(\mathcal{M})$

($\mathcal{M}$ pointed + functorial hofib)

- calculus

The n^{th} layer functor associated to T:

$$\Lambda^n : \mathcal{M}' \longrightarrow \mathcal{M} : x \mapsto hofib(T_n x \to T_{n-1} x).$$

Properties of calculi

$\Gamma : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$

- calculus

($\mathcal{M}$ pointed + functorial hofib)

The n^{th} layer functor associated to Γ:

$$\Lambda^n : \mathcal{M}' \longrightarrow \mathcal{M} : x \mapsto \mathrm{hofib}(\Gamma_n x \to \Gamma_{n-1} x).$$

- measures the difference between the degree $n-1$ and degree n approximations

Properties of calculi

$\Gamma : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$ - calculus

($\mathcal{M}$ pointed + functorial lwfib)

Lemma: Under reasonable conditions on Γ,

- $\Lambda_n x$ homogeneous, and
- $\Lambda_n y_n x : \Lambda_n x \xrightarrow{\sim} \Lambda_n \Gamma_n x \quad \forall x \in \mathrm{Ob}\mathcal{M}'$.

Properties of calculi

$T : \mathcal{M}' \to \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$

($\mathcal{M}$ pointed + functorial bofib)

- calculus

Lemma: Under reasonable conditions on T,

- $\Lambda_n x$ homogeneous, and
- $\Lambda_n y_n x : \Lambda_n x \xrightarrow{\sim} \Lambda_n T_n x \quad \forall x \in \mathrm{Ob}\mathcal{M}'$.

"The n^{th} layer of x is equal to the n^{th} layer of its degree n approx."

Example $\mathcal{M} = \mathrm{Top}$, usual WE

$\exists$ calculus $\Gamma : \mathrm{Top} \longrightarrow \mathrm{Tow}^{\mathbb{N}}(\mathrm{Top})$

with $\Gamma_n X = n^{\text{th}}$ Postnikov section of X,

i.e., $\Gamma X =$ Postnikov tower of X,

so $\Lambda_n X = K(\pi_n X, n)$.

Example: Ch_R = cat. of unbded ch cxes / R, equipped w/ proj. model structure

Example: Ch_R = cat. of unbded ch cxes / R, equipped w/ proj. model structure

$\exists$ calculus $\Gamma : Ch_R \to \mathrm{Tow}^{\mathbb{N}}(Ch_R)$ with

$$(\Gamma_n(X,d))_k = \begin{cases} X_k & : k \le n \\ X_{n+1}/\ker d_{n+1} & : k = n+1 \\ 0 & : \text{else} \end{cases} + \text{induced diff'l}$$

Example: Ch_R = cat. of unbded ch cxes / R, equipped w/ proj. model structure

$\exists$ calculus $T : Ch_R \to Tow^{\mathbb{N}}(Ch_R)$ with

$$(T_n(X,d))_k = \begin{cases} X_k & : k \le n \\ X_{n+1}/\ker d_{n+1} & : k = n+1 \\ 0 & : \text{else} \end{cases} \quad + \text{ induced diff'l}$$

$\gamma_n : Id_{Ch_R} \to T_n$, $\delta_n : T_n \to T_{n-1}$ quotient maps

Example: Ch_R = cat. of unbded ch cxes / R, equipped w/ proj. model structure

$\exists$ calculus $T : Ch_R \to \mathrm{Tow}^{\mathbb{N}}(Ch_R)$ with

$$H_k(\Gamma_n(X,d)) \cong \begin{cases} H_k(X,d) : k \leq n \\ 0 \quad : k > n . \end{cases}$$

Example: Ch_R = cat. of unbded ch cxes / R, equipped w/ proj. model structure

$\exists$ calculus $\Gamma : Ch_R \to \mathrm{Tow}^{\mathbb{N}}(Ch_R)$ with

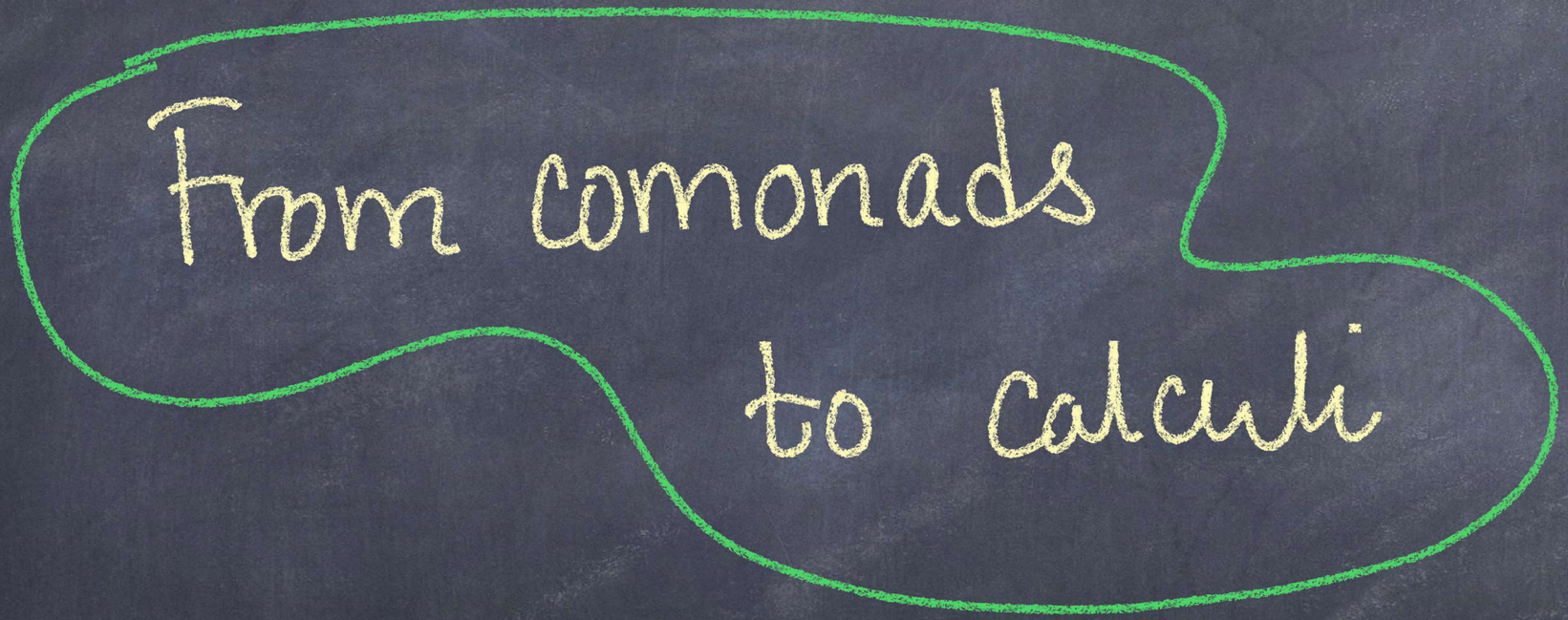
$$\Lambda_n(X,d) = \mathrm{hofib}\, \gamma_n(X,d) \simeq \begin{cases} \ker d_n : k = n \\ X_{n+1}/\ker d_{n+1} : k = n+1 \\ 0 : \text{else} \end{cases}$$

Example: Ch_R = cat. of unbded ch cxes / R, equipped w/ proj. model structure

$\exists$ calculus $\Gamma : Ch_R \to \mathrm{Tow}^{\mathbb{N}}(Ch_R)$ with

$$\Lambda_n(X,d) = \mathrm{hofib}\, \gamma_n(X,d) \simeq \begin{cases} \ker d_n : k=n \\ X_{n+1}/\ker d_{n+1} : k=n+1 \\ 0 : \text{else} \end{cases}$$

whence $$H_k(\Lambda_n(X,d)) \cong \begin{cases} H_k(X,d) : k=n \\ 0 : \text{else} \end{cases}.$$


From comonads
to calculi

Comonads

$\mathcal{A}$ - category

Comonad on $\mathcal{A}$ = comonoid in $(\mathcal{E}nd(\mathcal{A}), \circ)$

Comonads

$\mathcal{A}$ - category

Comonad on $\mathcal{A}$ = comonoid in $(\mathcal{E}nd(\mathcal{A}), \circ)$

$K : \mathcal{A} \to \mathcal{A}$

$\Delta : K \to K^2$

$\varepsilon : K \to Id_{\mathcal{A}}$

Comonads

$\mathcal{A}$ - category

Comonad on $\mathcal{A}$ = comonoid in $(\mathcal{E}nd(\mathcal{A}), \circ)$

$K : \mathcal{A} \to \mathcal{A}$

$\Delta : K \to K^2$ st

$\varepsilon : K \to Id_{\mathcal{A}}$

$$\begin{array}{ccc} K & \xrightarrow{\Delta} & K^2 \\ {\scriptstyle \Delta}\downarrow & \circlearrowright & \downarrow{\scriptstyle K\Delta} \\ K^2 & \xrightarrow[\Delta_K]{} & K^3 \end{array}$$

$$\begin{array}{ccc} K^2 & & \xrightarrow{K\varepsilon} \\ {\scriptstyle \Delta}\uparrow & \circlearrowright & \\ K & = & K \\ {\scriptstyle \Delta}\downarrow & \circlearrowright & \nearrow_{\varepsilon_K} \\ K^2 & & \end{array}$$

Comonads

$\mathcal{A}$ - category

$\mathbb{K} = (K, \Delta, \varepsilon)$, $\mathbb{K}' = (K', \Delta', \varepsilon')$ - comonads

Comonad morphism from $\mathbb{K}$ to $\mathbb{K}'$

= nat'l transf $\sigma: K \to K'$ st

Comonads

$\mathcal{A}$ - category

$\mathbb{K} = (K, \Delta, \varepsilon)$, $\mathbb{K}' = (K', \Delta', \varepsilon')$ - comonads

Comonad morphism from $\mathbb{K}$ to $\mathbb{K}'$

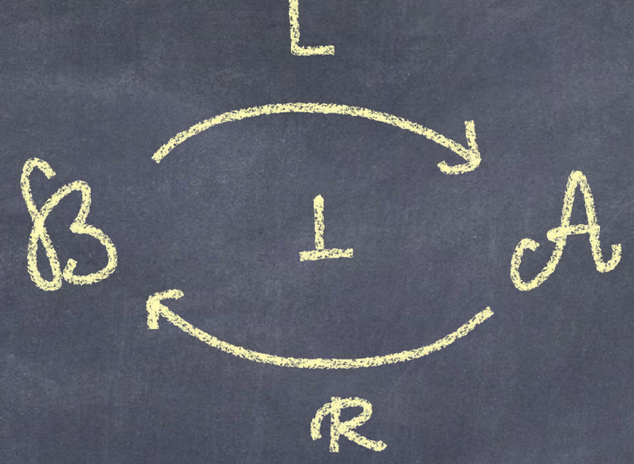
= nat'l transf $\sigma : K \to K$ st



$\Rightarrow \mathrm{Comon}_{\mathcal{A}}$

Comonads

Adjunction



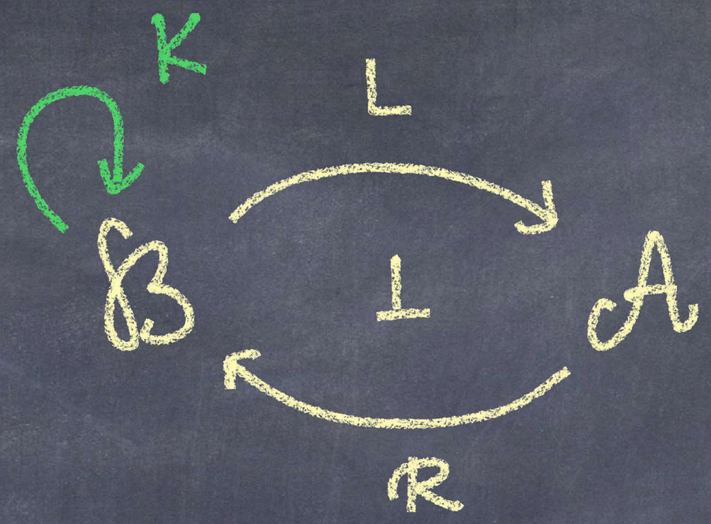
$$\begin{cases} \text{unit } \eta : Id_{\mathcal{B}} \to RL \\ \text{counit } \varepsilon : LR \to Id_{\mathcal{A}} \end{cases}$$

$\Rightarrow$ comonad on $\mathcal{A}$

$$(LR,\ L\eta_R,\ \varepsilon)$$

Comonads

Adjunction



$$\begin{cases} \text{unit } \eta : Id_{\mathcal{B}} \to RL \\ \text{counit } \varepsilon : LR \to Id_{\mathcal{A}} \end{cases}$$

$\Rightarrow$ comonad on $\mathcal{A}$

$$(LR,\ L\eta_R,\ \varepsilon)$$

Generalization : $\mathbb{K} = (K, \Delta, \tilde{\varepsilon})$ comonad on $\mathcal{B}$

$\Rightarrow \exists$ comonad on $\mathcal{A}$ $\quad (LKR, \dots, \dots)$.

Coalgebras $\mathbb{K} = (K, \Delta, \varepsilon)$ comonad on $\mathcal{A}$

$\mathbb{K}$-coalgebra $= c \in \mathrm{Ob}\,\mathcal{A} + \delta : c \to Kc$

Coalgebras $\mathbb{K} = (K, \Delta, \varepsilon)$ comonad on $\mathcal{A}$

$\mathbb{K}$-coalgebra $= c \in \mathrm{Ob}\,\mathcal{A} + \delta : c \to Kc$ st

$$\begin{array}{ccc} c & \xrightarrow{\delta} & Kc \\ {\scriptstyle\delta}\downarrow & \circlearrowright & \downarrow{\scriptstyle K\delta} \\ Kc & \xrightarrow[\Delta_c]{} & K^2c \end{array} \quad + \quad \begin{array}{ccc} c & \xrightarrow{\delta} & Kc \\ \| & \circlearrowright & \swarrow{\scriptstyle \varepsilon_c} \\ c & & \end{array}$$

$\Rightarrow$ category $\mathrm{Coalg}_{\mathbb{K}}$

The bar construction

$\mathbb{K} = (K, \Delta, \varepsilon)$ - comonad on $\mathcal{A}$

$$\mathrm{Bar}_\bullet^{\mathbb{K}} : \mathcal{A} \longrightarrow \mathcal{A}^{\Delta^{op}}$$

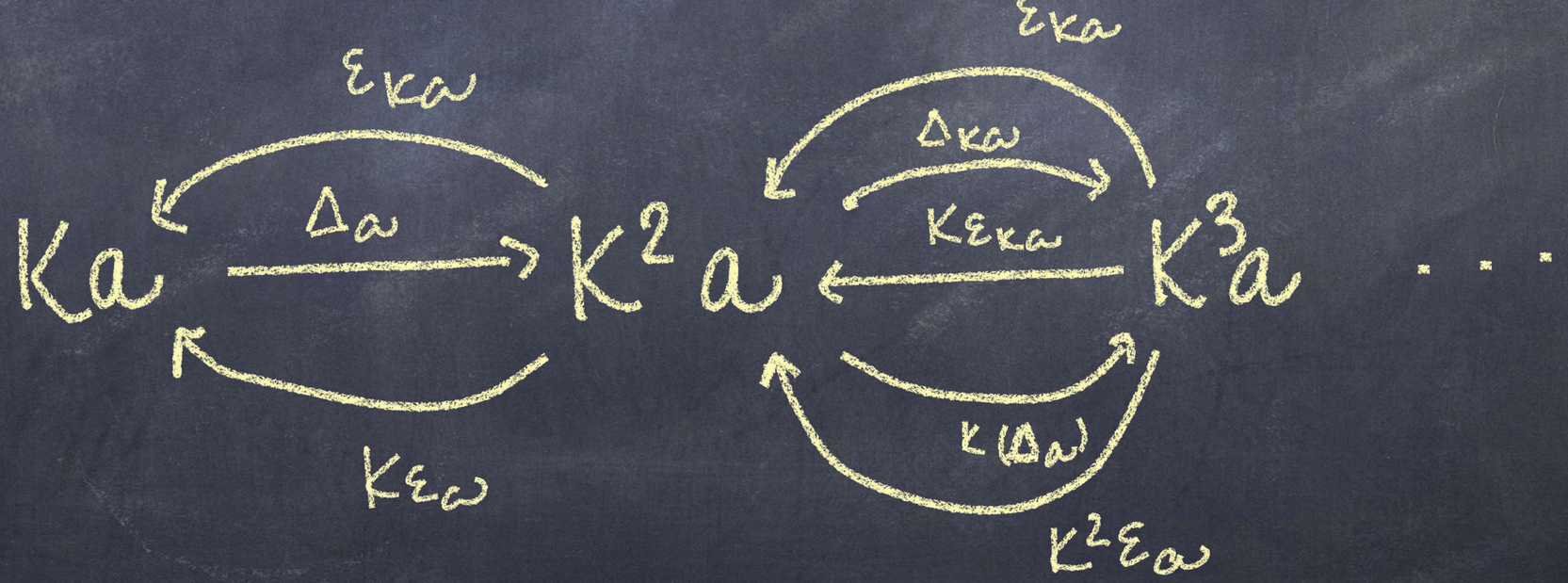
$$\mathrm{Bar}_n^{\mathbb{K}}(a) = K^{n+1}a$$

The bar construction

$\mathbb{K} = (K, \Delta, \varepsilon)$ - comonad on $\mathcal{A}$

$$\mathrm{Bar}_\bullet^{\mathbb{K}} : \mathcal{A} \longrightarrow \mathcal{A}^{\Delta^{op}}$$

$$\mathrm{Bar}_n^{\mathbb{K}}(a) = K^{n+1}a$$

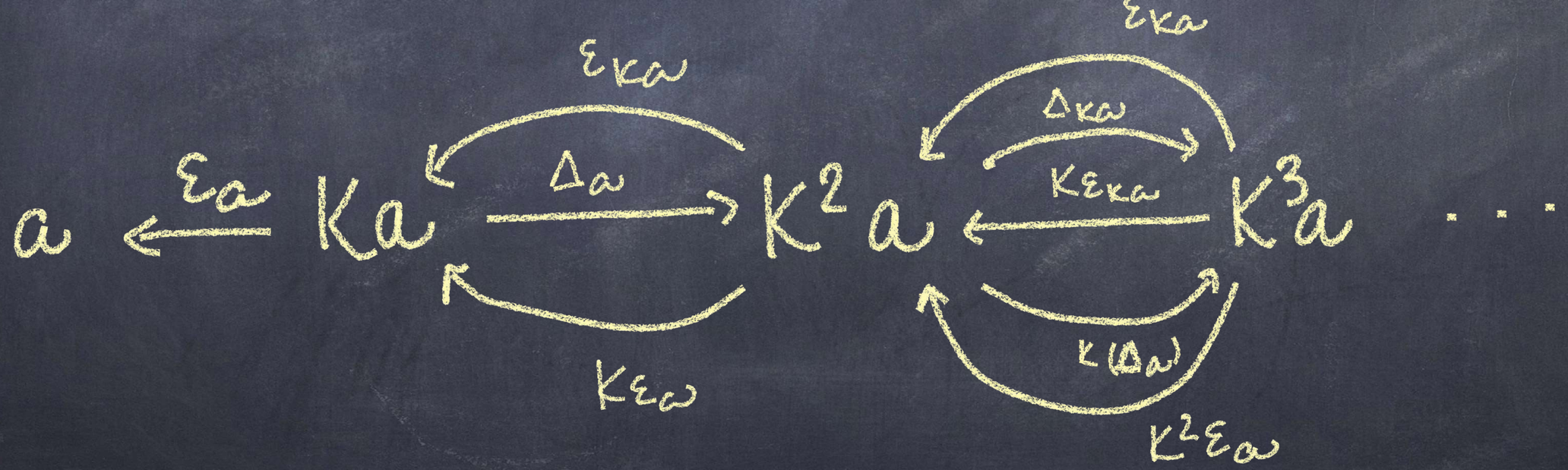


The bar construction

$\mathbb{K} = (K, \Delta, \varepsilon)$ - comonad on $\mathcal{A}$

$$\mathrm{Bar}_\bullet^{\mathbb{K}} : \mathcal{A} \longrightarrow \mathcal{A}^{\Delta^{op}}$$

$$\mathrm{Bar}_n^{\mathbb{K}}(a) = K^{n+1}a$$

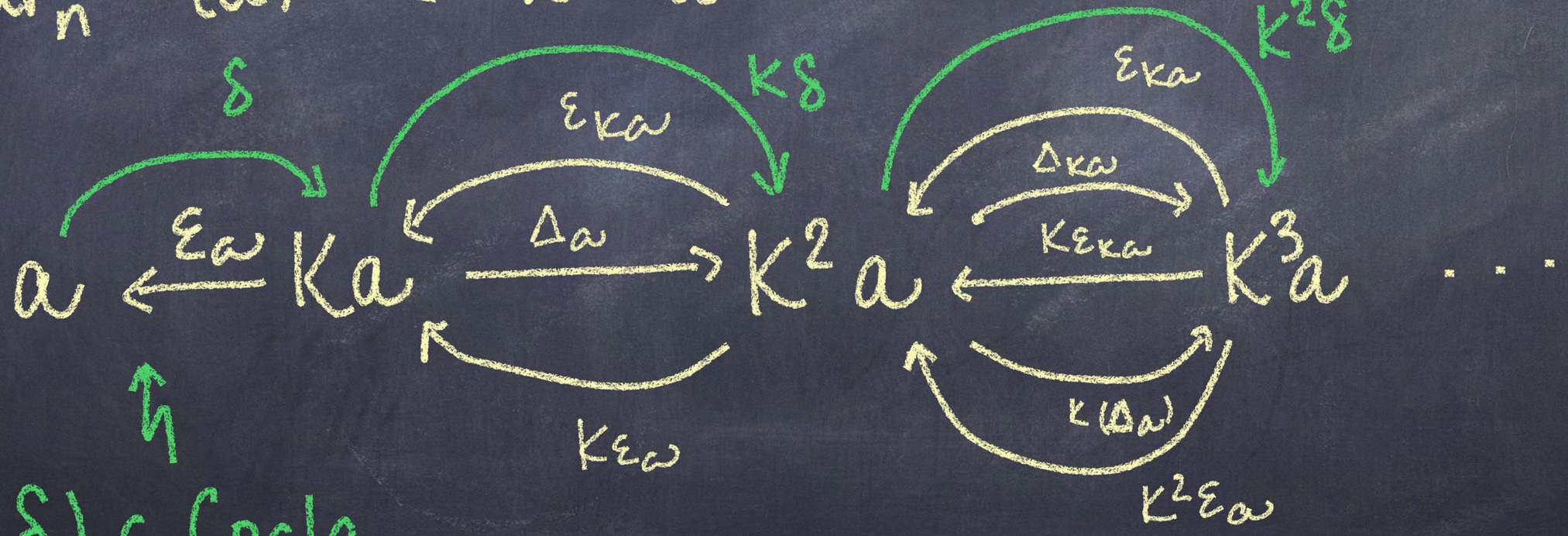


The bar construction

$\mathbb{K} = (K, \Delta, \varepsilon)$ - comonad on $\mathcal{A}$

$$\mathrm{Bar}_\bullet^{\mathbb{K}} : \mathcal{A} \longrightarrow \mathcal{A}^{\Delta^{op}}$$

$$\mathrm{Bar}_n^{\mathbb{K}}(a) = K^{n+1}a$$



$(a, \delta) \in \mathrm{Coalg}_{\mathbb{K}}$

The bar construction

$\mathcal{A}$ - category

$\exists$ bifunctor

$$\mathrm{Bar} : \mathrm{Comon}_{\mathcal{A}} \times \mathcal{A} \longrightarrow \mathcal{A}^{\Delta^{op}}$$

$$(\mathbb{K}, a) \longmapsto \mathrm{Bar}_{\cdot}^{\mathbb{K}}(a).$$

First step towards calculus

$\mathcal{M}$ - pted, simplicial model category

$\exists$ functor $\Gamma : \mathrm{Comon}_{\mathcal{M}} \times \mathcal{M} \longrightarrow \mathcal{M}$

$$\Gamma_{/K}(X) = \mathrm{hocofib}\left(|\mathrm{Bar}_{\cdot}^{/K} X| \xrightarrow{\varepsilon} X \right)$$

First step towards calculus

$\mathcal{M}$ - pted, simplicial model category

$\exists$ functor $\Gamma : \mathrm{Comon}_{\mathcal{M}} \times \mathcal{M} \longrightarrow \mathcal{M}$

$$\Gamma_{\mathbb{K}}(x) = \mathrm{hocofib}\left(|\mathrm{Bar}_{\cdot}^{\mathbb{K}} x| \xrightarrow{\varepsilon} x \right)$$

ε: weak equiv if x admits $\mathbb{K}$-coalg structure

First step towards calculus

$\mathcal{M}$ - pted, simplicial model category

$\exists$ functor $\Gamma : \mathrm{Comon}_{\mathcal{M}} \times \mathcal{M} \longrightarrow \mathcal{M}$

$$\Gamma_{\mathbb{K}}(x) = \mathrm{hocofib}\left(|\mathrm{Bar}_{\bullet}^{\mathbb{K}} x| \xrightarrow{\varepsilon} x \right)$$

Rmk: $\forall x$, $\exists$ natural $y_{\mathbb{K}}(x) : x \longrightarrow \Gamma_{\mathbb{K}}(x)$

Nice comonads

$\mathcal{M}$ pted simplicial model category

$\mathbb{K} \in \mathrm{Comon}_{\mathcal{M}}$ is compliant if:

- $\mathrm{Bar}_\cdot^{\mathbb{K}}$ takes values in Reedy cofibrant objects
- K preserves weak equivalences
- $\mathrm{colim}_{\Delta^{op}} K X_\cdot \xrightarrow{\sim} K(\mathrm{colim}_{\Delta^{op}} X_\cdot)$

$\forall\, X_\cdot \in \mathcal{M}^{\Delta^{op}}$.

Towers of comonads create tower functors

$\mathcal{M}$ - pted simplicial model cat

$\Rightarrow \exists \quad T_{\mathcal{M}} : \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}}) \times \mathcal{M} \longrightarrow \mathrm{Tow}^{n}(\mathcal{M})$

Towers of comonads create tower functors

$\mathcal{M}$ - pted simplicial model cat

$\Rightarrow \exists \quad T_{\mathcal{M}} : \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}}) \times \mathcal{M} \longrightarrow \mathrm{Tow}^{\eta}(\mathcal{M})$

$$\mathcal{K} = \cdots \longrightarrow K_n \xrightarrow{\sigma_n} K_{n-1} \longrightarrow \cdots \xrightarrow{\sigma_1} K_0$$

Towers of comonads create tower functors

$\mathcal{M}$ - pted simplicial model cat

$$\Rightarrow \exists \quad \Pi_{\mathcal{M}} : \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}}) \times \mathcal{M} \longrightarrow \mathrm{Tow}^{\eta}(\mathcal{M})$$

$$\mathcal{K} = \cdots \longrightarrow |K_n| \xrightarrow{\sigma_n} |K_{n-1}| \longrightarrow \cdots \xrightarrow{\sigma_1} |K_0|$$

$$\Rightarrow \pi_{\mathcal{M}}(\mathcal{K}, X) =$$

$$\cdots \longrightarrow \Gamma_{|K_n|}(X) \xrightarrow{\Gamma_{\sigma_n}(X)} \Gamma_{|K_{n-1}|}(X) \longrightarrow \cdots$$

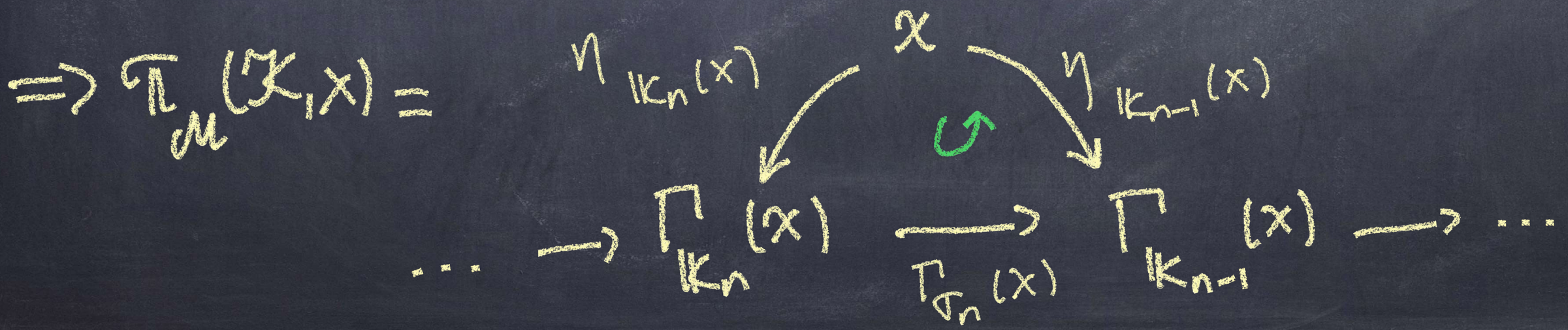
Towers of comonads create tower functors

$\mathcal{M}$ - ptd simplicial model cat

$\Rightarrow \exists \quad \Pi_{\mathcal{M}} : \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}}) \times \mathcal{M} \longrightarrow \mathrm{Tow}^{\eta}(\mathcal{M})$

$$\mathcal{K} = \cdots \longrightarrow |K_n| \xrightarrow{\sigma_n} |K_{n-1}| \longrightarrow \cdots \xrightarrow{\sigma_1} |K_0|$$

$\Rightarrow \pi_{\mathcal{M}}(\mathcal{K}, x) =$



Towers of comonads create calculi

$\mathcal{M}$ - pted simplicial model cat, $\mathcal{M}' \subseteq \mathcal{M}$

$\mathcal{K} = \ldots \to \mathbb{K}_n \xrightarrow{\nabla_n} \mathbb{K}_{n-1} \to \ldots \xrightarrow{\nabla_1} \mathbb{K}_0$

- tower of compliant comonads

Towers of comonads create calculi

$\mathcal{M}$ - pted simplicial model cat, $\mathcal{M}' \subseteq \mathcal{M}$

$\mathcal{K} = \ldots \to \mathbb{K}_n \xrightarrow{\tau_n} \mathbb{K}_{n-1} \to \ldots \xrightarrow{\tau_1} \mathbb{K}_0$

- tower of compliant comonads

$\Rightarrow$ $\mathbb{T}_{\mathcal{M}}(\mathcal{K}, -) : \mathcal{M}' \to \mathrm{Tow}^{\mathrm{op}}(\mathcal{M})$

is a calculus.

Homogeneity

If moreover $\mathcal{M}$ stable and all objects fibrant and each K_n commutes up to htpy with holims, then $\Lambda^n(T_{\mathcal{M}}(\mathcal{F}, X))$ is n-homogeneous $\forall n$.

"Example" $\mathcal{M} = Ch_R$

$\forall n > 0, \exists \quad Ch_R^{\geq n} \underset{\rho_n}{\overset{2n}{\rightleftarrows}} Ch_R$ (with ι)

$$\rho_n(X,d) = \begin{cases} 0 : k < n \\ \ker d_n : k = n \\ X_k : k > n \end{cases}$$

"Example" $\mathcal{M} = Ch_R$

$\forall n > 0, \exists$ $Ch_R^{\geq n} \underset{\rho_n}{\overset{2n}{\rightleftarrows}} Ch_R$ $\Big\}$ $\Rightarrow$ comonad $\mathbb{K}_n$

$$\rho_n(X,d) = \begin{cases} 0 : k < n \\ \ker d_n : k = n \\ X_k : k > n \end{cases}$$

"Example" $\mathcal{M} = Ch_R$

$\forall n > 0, \exists$ $Ch_R^{\mathbb{Z}/n} \underset{\rho_n}{\overset{2_n}{\rightleftarrows}} Ch_R$ $\Big\}$ $\Rightarrow$ comonad $\mathbb{K}_n$

$\Rightarrow$ tower of comonads $\cdots \to \mathbb{K}_n \xrightarrow{\tau_n} \mathbb{K}_{n-1} \to \cdots$

$= \mathcal{K}$

"Example" $\mathcal{M} = Ch_R$

$\forall n > 0$, $\exists$ $Ch_R^{\geq n} \underset{\rho_n}{\overset{2n}{\rightleftarrows}} Ch_R$ $\Big\}$ $\Rightarrow$ comonad $\mathbb{K}_n$

$\Rightarrow$ tower of comonads $\cdots \to \mathbb{K}_n \xrightarrow{\nabla_n} \mathbb{K}_{n-1} \to \cdots$

$\Rightarrow$ calculus $T_{Ch_R}(\mathbb{K}, -) : Ch_R \to Tow^{\mathbb{N}}(Ch_R)$

"Example" $\mathcal{M} = Ch_R$

$\forall n > 0, \exists$ $Ch_R^{\mathbb{Z}/n} \underset{\rho_n}{\overset{2n}{\rightleftarrows}} Ch_R$ $\Big\}$ $\Rightarrow$ comonad $\mathbb{K}_n$

$\Rightarrow$ tower of comonads $\cdots \to \mathbb{K}_n \xrightarrow{\tau_n} \mathbb{K}_{n-1} \to \cdots$

$\Rightarrow$ calculus $\mathcal{T}_{Ch_R}(\mathbb{K}_1, -) : Ch_R \to Tow^{\eta}(Ch_R)$

- exactly the calculus defined previously!

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

Given $\mathbb{P} = \{p_k \mid k \in \mathbb{N}\}$ w/ p_k prime $\forall k$,

let $R_n = \mathbb{Z}\left[\frac{1}{p_k} \mid 1 \leq k \leq n\right]$.

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

Given $\mathcal{P} = \{p_k \mid k \in \mathbb{N}\}$ w/ p_k prime $\forall k$,

let $R_n = \mathbb{Z}\left[\frac{1}{p_k} \mid 1 \leq k \leq n\right]$.

The unit $\eta_n : \mathbb{Z} \hookrightarrow R_n$ induces

$$\mathrm{Mod}_{R_n}^{\Delta^{op}} \underset{\mathrm{Hom}(R_n, -)}{\overset{\eta_n^*}{\rightleftarrows}} \mathcal{M}$$

, associated comonad $\mathbb{K}_n$ on $\mathcal{M}$.

Example $\mathcal{M} = \mathcal{M}_0^{\Delta^{op}}$ w/ usual simplicial model structure

$\mathrm{Mod}_{R_n}^{\Delta^{op}}$ 

$\mathcal{M}$, associated comonad $\mathbb{K}_n$ on $\mathcal{M}$.

$R_n \hookrightarrow R_{n+1} \Rightarrow \nabla_n : \mathbb{K}_{n+1} \to \mathbb{K}_n$

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

$\mathrm{Mod}_{R_n}^{\Delta^{op}}$ $\underset{\mathrm{Hom}(R_n,-)}{\overset{2_n^*}{\rightleftarrows}}$ $\mathcal{M}$, associated comonad $\mathbb{K}_n$ on $\mathcal{M}$.

$R_n \hookrightarrow R_{n+1} \Rightarrow \nabla_n : \mathbb{K}_{n+1} \to \mathbb{K}_n$

$\Rightarrow \mathcal{K}_{TP} = (\cdots \to \mathbb{K}_n \xrightarrow{\nabla_n} \mathbb{K}_{n-1} \to \cdots) \in \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}})$.

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

$$\mathcal{K}_{TP} = (\cdots \to |K_n| \xrightarrow{r_n} |K_{n-1}| \to \cdots) \in \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}})$$

$$\mathcal{M}' = \{A_\cdot \in \mathrm{Ob}\,\mathcal{M} \mid A_n \text{ torsion-free } \forall n \geq 0\}$$

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

$$\mathcal{K}_{TP} = (\cdots \to |K_n \xrightarrow{r_n} |K_{n-1} \to \cdots) \in \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}})$$

$$\mathcal{M}' = \{A_\cdot \in \mathrm{Ob}\,\mathcal{M} \mid A_n \text{ torsion-free } \forall n \geq 0\}$$

$$\Rightarrow \pi_{\mathcal{M}}(\mathcal{K}_P, -) : \mathcal{M}' \to \mathrm{Tow}^{\gamma}(\mathcal{M})$$

is a calculus.

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

$$\pi_{\mathcal{M}}(\mathcal{K}_{\mathbb{P}}, -) : \mathcal{M}' \longrightarrow \mathrm{Tow}'(\mathcal{M})$$

↖ The "$\mathbb{P}$-divisibility calculus"

Example $\mathcal{M} = \mathcal{A}b^{\Delta^{op}}$ w/ usual simplicial model structure

$$\pi_{\mathcal{M}}(\mathcal{K}_{\mathbb{P}}, -) : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\mathbb{N}}(\mathcal{M})$$

↖ The "$\mathbb{P}$-divisibility calculus"

$\Rightarrow \Lambda n\left(\pi_{\mathcal{M}}(\mathcal{K}_{\mathbb{P}}, A.)\right)$

– detects the part of $A.$ that is p_n-divisible but not p_k-divisible, for $k < n$.

Summary $\mathcal{M}$ - ptd simplicial model cat
$\mathcal{M}' \subseteq \mathcal{M}$

- $\exists\ T_{\mathcal{M}} : \mathrm{Tow}(\mathrm{Comon}_{\mathcal{M}}) \times \mathcal{M} \longrightarrow \mathrm{Tow}^{\eta}(\mathcal{M})$
- If $\mathcal{K}$ tower of compliant comonads, then

$$T_{\mathcal{M}}(\mathcal{K}, -) : \mathcal{M}' \longrightarrow \mathrm{Tow}^{\eta}(\mathcal{M})$$

is a calculus.

From $\mathcal{P}(n)$-modules

to comonads

$\mathcal{P}(n)$ and its modules

$\mathcal{P}(n)$ = poset of subsets of $\{1, \dots, n\}$, with strict monoidal structure $\cup$

$\mathcal{P}(n)$ and its modules

$\mathcal{P}(n)$ = poset of subsets of $\{1, \dots, n\}$, with strict monoidal structure $\cup$

$n=1$: $\emptyset \longrightarrow \{1\}$

$n=2$:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \{2\} & \longrightarrow & \{1,2\} \end{array}$$

etc. !

$\mathcal{P}(n)$ and its modules

$\mathcal{P}(n)$ = poset of subsets of $\{1, \dots, n\}$, with strict monoidal structure $\cup$

Functors $F: \mathcal{P}(n) \longrightarrow \mathcal{A}$ correspond to n-cubical diagrams in $\mathcal{A}$.

$\mathcal{P}(n)$ and its modules

$\mathcal{P}(n)$ = poset of subsets of $\{1, \dots, n\}$, with strict monoidal structure $\cup$

$\mathcal{P}(n)$-module = category $\mathcal{O}$ + bifunctor $\Theta : \mathcal{O} \times \mathcal{P}(n) \to \mathcal{O}$ such that $\Theta(-, \emptyset) = \mathrm{Id}_{\mathcal{O}}$ and $\Theta(\Theta(-, S), T) = \Theta(-, S \cup T)$.

$\Rightarrow$ category $\mathrm{Mod}_{\mathcal{P}(n)}$

Characterization $\mathscr{D}$ - category

TFAE:

- $\mathcal{P}(n)$ - module structure on $\mathscr{D}$;
- $\{\mathbb{M}_1, \ldots, \mathbb{M}_n\}$ - strict idempotent monads on $\mathscr{D}$ such that $M_i M_j = M_j M_i \ \forall i, j$;

Characterization $\mathcal{D}$ - category

TFAE:

- $\mathcal{P}(n)$ - module structure on $\mathcal{D}$;
- $\{\mathbb{M}_1, \dots, \mathbb{M}_n\}$ - strict idempotent monads on $\mathcal{D}$ such that $M_i M_j = M_j M_i \ \forall i, j$;
- $\{\mathcal{D} \underset{I}{\overset{L_j}{\rightleftarrows}} \mathcal{D}_j \mid 1 \le j \le n\}$ st $\mathcal{D} \xrightarrow{L_j} \mathcal{D}_j$

  $L_i \downarrow \circlearrowright \downarrow L_i|_{\mathcal{D}_j}$

  $\mathcal{D}_i \xrightarrow[L_j|_{\mathcal{D}_i}]{} \mathcal{D}_i \cap \mathcal{D}_j$

↑ localization

Characterization $\mathcal{D}$ - category

Idea :

$\mathcal{D} \times \mathcal{P}(n) \xrightarrow{\mathcal{O}} \mathcal{D}$

$\Longleftrightarrow$

$\mathcal{P}(n) \xrightarrow{\mathcal{O}^{\#}} End(\mathcal{D})$

n-cubical diagram of endofunctors

Characterization $\mathcal{D}$ - category

Idea :

n-cubical diagram of endofunctors

$$\mathcal{D} \times \mathcal{P}(n) \xrightarrow{\Theta} \mathcal{D} \qquad \mathcal{P}(n) \xrightarrow{\Theta^{\#}} End(\mathcal{D})$$

$$\Theta(-, \phi) = Id_{\mathcal{D}} \iff \Theta^{\#}(\phi) = Id_{\mathcal{D}}$$

Characterization $\mathcal{D}$ - category

Idea :

$\mathcal{D} \times \mathcal{P}(n) \xrightarrow{\Theta} \mathcal{D}$

$\Theta(-, \phi) = Id_{\mathcal{D}}$

$\Theta((-,S),T) = \Theta(-, S \cup T)$

$= \Theta((-,T),S)$

$\Longleftrightarrow$

n-cubical diagram of endofunctors

$\mathcal{P}(n) \xrightarrow{\Theta^{\#}} End(\mathcal{D})$

$\Theta^{\#}(\phi) = Id_{\mathcal{D}}$

$\Theta^{\#}(T) \circ \Theta^{\#}(S)$

$= \Theta^{\#}(S \cup T)$

$= \Theta^{\#}(S) \circ \Theta^{\#}(T)$

"Example" R commutative ring

$\mathcal{S} = \{S_1, \ldots, S_n\}$ w/ $S_j \subseteq R$ $\forall j$.

$$\mathcal{O}_{\mathcal{S}} : \mathrm{Mod}_R \times \mathcal{P}(n) \longrightarrow \mathrm{Mod}_R$$

$$(M, T) \mapsto M \underset{R}{\otimes} R\left[\left(\bigcup_{j \in T} S_j\right)^{-1}\right]$$

"Example" R commutative ring

$\Sigma = \{S_1, \dots, S_n\}$ w/ $S_j \subseteq R$ $\forall j$.

$$\mathcal{O}_\Sigma : \mathrm{Mod}_R \times \mathcal{P}(n) \longrightarrow \mathrm{Mod}_R$$

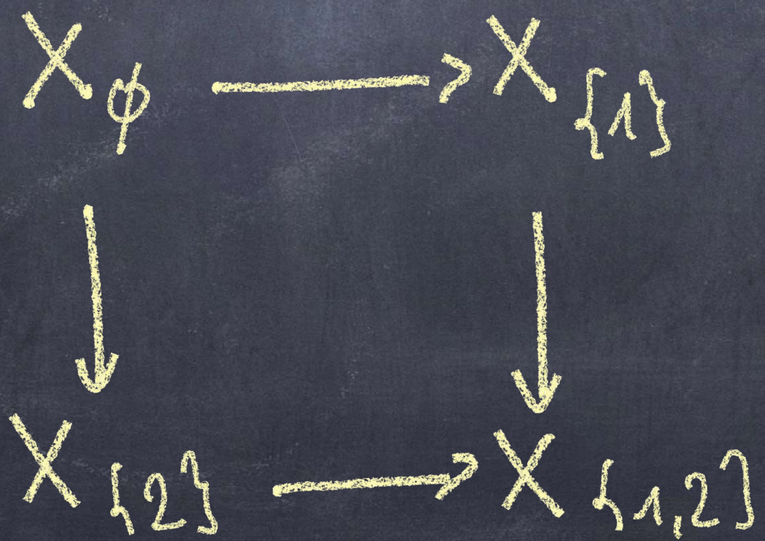
$$(M, T) \mapsto M \otimes_R R\left[\left(\bigcup_{j \in T} S_j\right)^{-1}\right]$$

$$\iff \{\ \mathrm{Mod}_R \underset{\perp}{\overset{L_j}{\rightleftarrows}} \mathrm{Mod}_{R[S_j^{-1}]} \mid 1 \le j \le n \}$$

The first step towards comonads

$\mathcal{M}$ pted simplicial model category

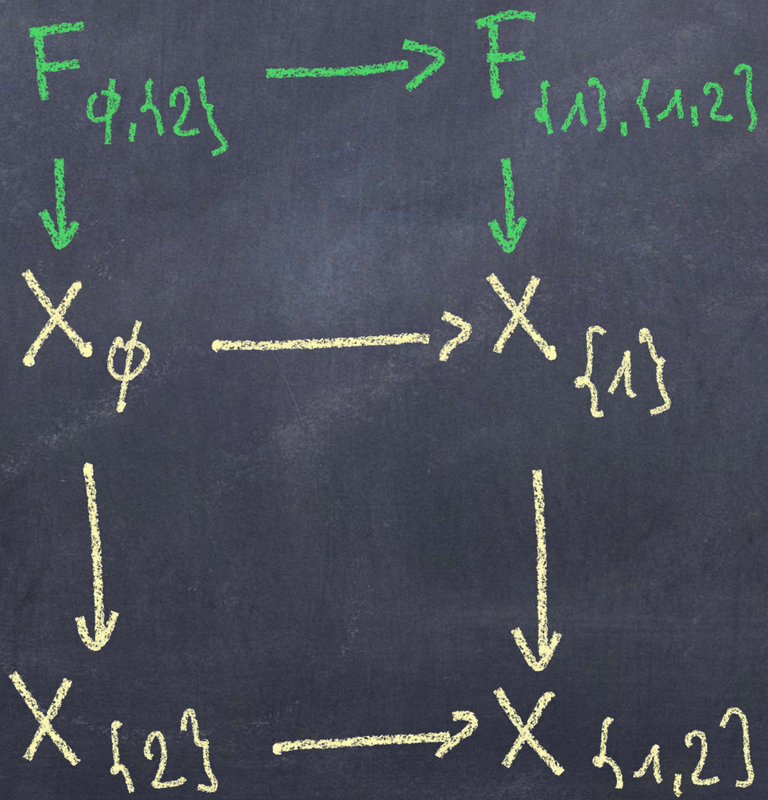
The iterated fiber of a square $\mathcal{X} : \mathcal{P}(2) \to \mathcal{M}$



The first step towards comonads

$\mathcal{M}$ pted simplicial model category

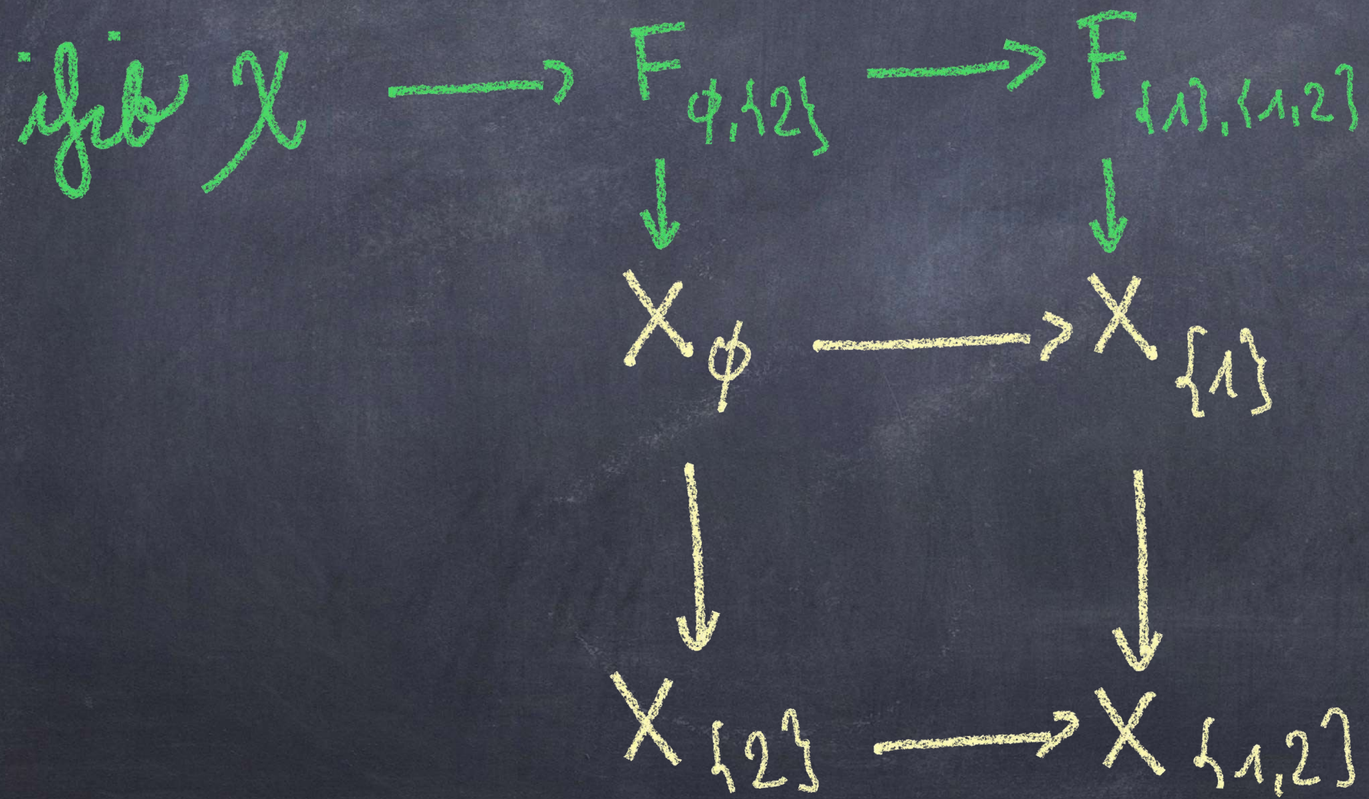
The iterated fiber of a square $\mathcal{X} : \mathcal{P}(2) \to \mathcal{M}$



The first step towards comonads

$\mathcal{M}$ pted simplicial model category

The iterated fiber of a square $\mathcal{X} : \mathcal{P}(2) \to \mathcal{M}$



The first step towards comonads

$\mathcal{M}$ pted simplicial model category

The iterated fiber of a square $\mathcal{X} : \mathcal{P}(2) \to \mathcal{M}$

$$\mathrm{ifib}\,\mathcal{X} \simeq \mathrm{holim}\left(\begin{array}{ccccc} X_{\emptyset} & \longrightarrow & X_{\{1\}} & \longleftarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X_{\{2\}} & \longrightarrow & X_{\{1,2\}} & \longleftarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ * & \longrightarrow & * & \longleftarrow & * \end{array}\right)$$

The first step towards comonads

$\mathcal{M}$ pted simplicial model category

The iterated fiber of an n-cube $\mathcal{X}: \mathcal{P}(n) \to \mathcal{M}$

$$\mathrm{ifib}\, \mathcal{X} \simeq \operatorname*{holim}_{\mathcal{V}^{\times n}} (\varphi_n)_! (\mathcal{X})$$

$\mathcal{V} = (0 \xrightarrow{t} 1 \xleftarrow{e} v)$, $\varphi_n : \mathcal{P}(n) \longrightarrow \mathcal{V}^{\times n}$

$$\varphi_n(\tau)_i = \begin{cases} 0 : i \notin \tau \\ 1 : i \in \tau \end{cases}.$$

The n-cubes of interest

$\mathcal{M}$ pted simplicial model category

$\mathcal{O} : \mathcal{D} \times \mathcal{P}(n) \to \mathcal{D}$ $\mathcal{P}(n)$-module

$F \in \mathrm{Fun}(\mathcal{D}, \mathcal{M})$

The n-cubes of interest

$\mathcal{M}$ pted simplicial model category

$\mathcal{O} : \mathcal{D} \times \mathcal{P}(n) \to \mathcal{D}$ $\mathcal{P}(n)$-module

$F \in \mathrm{Fun}(\mathcal{D}, \mathcal{M})$

$\Rightarrow \forall d \in \mathrm{Ob}\,\mathcal{D}$, $\exists$ $\mathcal{X}_d^{\mathcal{O},F} : \mathcal{P}(n) \to \mathcal{M}$

$T \mapsto F(\mathcal{O}(d,T))$.

$n=2$:

$$\mathcal{D} \times \mathcal{P}(2) \xrightarrow{\Theta} \mathcal{D} \xrightarrow{F} \mathcal{M}$$

$$(d, S) \longmapsto d \cdot S \mapsto F(d \cdot S)$$

$$\Rightarrow \chi_d^{\Theta, F} = \begin{pmatrix} F(d \cdot \emptyset) & \longrightarrow & F(d \cdot \{1\}) \\ \downarrow & & \downarrow \\ F(d \cdot \{2\}) & \longrightarrow & F(d \cdot \{1,2\}) \end{pmatrix}$$

$\mathcal{P}(n)$-modules induce comonads

$\mathcal{M}$ pointed simplicial model category

$\Rightarrow \exists \quad \mathbb{K}^{\mathcal{M}}_n : \mathrm{Mod}^{op}_{\mathcal{P}(n)} \longrightarrow \mathrm{Comon}$

$\mathcal{P}(n)$-modules induce comonads

$\mathcal{M}$ pointed simplicial model category

$\Rightarrow \exists \; \mathbb{K}^{\mathcal{M}}_n : \mathrm{Mod}^{op}_{\mathcal{P}(n)} \longrightarrow \mathrm{Comon}$

st $\mathbb{K}^{\mathcal{M}}_n(\mathcal{O}, \Theta)$ comonad on $\mathrm{Fun}(\mathcal{O}, \mathcal{M})$

$\mathcal{P}(n)$-modules induce comonads

$\mathcal{M}$ pointed simplicial model category

$\Rightarrow \exists$ $\mathbb{K}^{\mathcal{M}}_n : \mathrm{Mod}^{op}_{\mathcal{P}(n)} \longrightarrow \mathrm{Comon}$

st $\mathbb{K}^{\mathcal{M}}_n(\mathcal{D}, \Theta)$ comonad on $\mathrm{Fun}(\mathcal{D}, \mathcal{M})$

w/ underlying endofunctor $K^{\mathcal{M}}_\Theta$ st

$K^{\mathcal{M}}_\Theta(F)(d) = \mathrm{ifib}\, \mathcal{X}^{\Theta,F}_d$ $\forall$ $F : \mathcal{D} \to \mathcal{M}$, $d \in \mathrm{Ob}\mathcal{D}$.

A relative version

$\mathcal{C}$ - category

$\mathcal{C}\mathrm{Mod}_{\mathcal{P}(n)}$

module structure

$(\mathcal{C} \xrightarrow{T} \mathcal{O},\ \mathcal{D} \times \mathcal{P}(n) \xrightarrow{\otimes} \mathcal{D})$

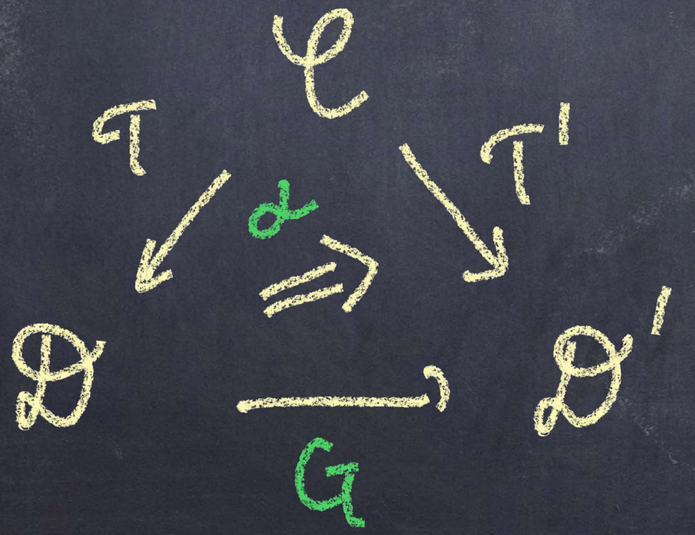
A relative version

$\mathcal{C}$ - category

$\mathcal{C}\,\mathrm{Mod}_{\mathcal{P}(n)}$

module structure

$$(\mathcal{C} \xrightarrow{T} \mathcal{D},\ \mathcal{D} \times \mathcal{P}(n) \xrightarrow{\otimes} \mathcal{D})$$



A relative version

$\mathcal{C}$ - category

$\mathcal{C}\,\mathrm{Mod}_{\mathcal{P}(n)}$

module structure

$$(\mathcal{C} \xrightarrow{T} \mathcal{D},\ \mathcal{D}\times\mathcal{P}(n) \xrightarrow{\Theta} \mathcal{D})$$

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle T}\downarrow & \overset{\alpha}{\Rightarrow} & \searrow{\scriptstyle T'} \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{D}' \end{array} \quad + \quad \begin{array}{ccccc} \mathcal{C}\times\mathcal{P}(n) & & \xrightarrow{T'\times \mathrm{Id}} & & \mathcal{D}'\times\mathcal{P}(n) \\ \downarrow{\scriptstyle T\times \mathrm{Id}} & & \overset{\beta}{\swarrow} & & \downarrow{\scriptstyle \Theta'} \\ \mathcal{D}\times\mathcal{P}(n) & \xrightarrow{\Theta} & \mathcal{D} & \xrightarrow{G} & \mathcal{D}' \end{array}$$

A relative version $\mathcal{C}$ - small category

For every pted simplicial model cat $\mathcal{M}$,

$\exists$ $K_n^{\mathcal{C},\mathcal{M}} : {}_{\mathcal{C}}\mathrm{Mod}_{\mathcal{P}(n)}^{op} \longrightarrow \mathrm{Comon}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$

A relative version $\mathcal{C}$ - small category

For every ptd simplicial model cat $\mathcal{M}$,

$$\exists \quad \mathbb{K}_n^{\mathcal{C},\mathcal{M}} : {}_{\mathcal{C}}\mathrm{Mod}_{\mathcal{P}(n)}^{op} \longrightarrow \mathrm{Comon}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$$

such the endofunctor $\mathbb{K}_r^{\mathcal{C},\mathcal{M}}(T,\Theta)$ is

$$\mathrm{Fun}(\mathcal{C},\mathcal{M}) \xrightarrow{T^*} \mathrm{Fun}(\Theta,\mathcal{M}) \xrightarrow{K_\Theta^{\mathcal{M}}} \mathrm{Fun}(\Theta,\mathcal{M}) \xrightarrow{T^*} \mathrm{Fun}(\mathcal{C},\mathcal{M}).$$

Towers of relative modules $\mathcal{C}$ - small category

$$\Big((\mathcal{D}_n, \Theta_n),\ \mathcal{C} \xrightarrow{T_n} \mathcal{D}_n,\ (T_n, \Theta_n) \xrightarrow{(G_n, \alpha_n, \beta_n)} (T_{n+1}, \ell_n^* \Theta_{n+1}) \Big)_{n \geq 0}$$

Towers of relative modules $\mathcal{C}$ - small category

$$\Big((\mathcal{D}_n, \Theta_n),\ \mathcal{C} \xrightarrow{T_n} \mathcal{D}_n,\ (T_n, \Theta_n) \xrightarrow{(G_n, \alpha_n, \beta_n)} (T_{n+1}, \ell_n^* \Theta_{n+1}) \Big)_{n \geq 0}$$

$\mathcal{D}_n$ ↖ $\mathcal{P}(n)$ - module

Towers of relative modules $\mathcal{C}$ - small category

$$\Big((\mathcal{D}_n, \Theta_n),\ \mathcal{C} \xrightarrow{T_n} \mathcal{D}_n,\ (T_n, \Theta_n) \xrightarrow{(G_n, \alpha_n, \beta_n)} (T_{n+1}, \ell_n^* \Theta_{n+1}) \Big)_{n \geq 0}$$

T_n ← functor

$(\mathcal{D}_n, \Theta_n)$ ↖ $\mathcal{P}(n)$ - module

Towers of relative modules

$\mathcal{C}$ - small category

$$\Big((\mathcal{D}_n, \Theta_n),\ \mathcal{C} \xrightarrow{T_n} \mathcal{D}_n,\ (\tau_n, \theta_n) \xrightarrow{(G_n, \alpha_n, \beta_n)} (\tau_{n+1}, \varrho_n^* \theta_{n+1}) \Big)_{n \geq 0}$$

T_n: functor

$(\mathcal{D}_n, \Theta_n)$: $\mathcal{P}(n)$ - module

(G_n, α_n, β_n): morphism in $\mathcal{C}\mathrm{Mod}_{\mathcal{P}(n)}$

Towers of relative modules

$\mathcal{C}$ – small category

$\iota_n : \mathcal{P}(n) \hookrightarrow \mathcal{P}(n+1)$

$$\Big((\mathcal{D}_n, \Theta_n),\ \mathcal{C} \xrightarrow{T_n} \mathcal{D}_n,\ (T_n, \Theta_n) \xrightarrow{(G_n, \alpha_n, \beta_n)} (T_{n+1}, \iota_n^* \Theta_{n+1}) \Big)_{n \geq 0}$$

T_n: functor

$(\mathcal{D}_n, \Theta_n)$: $\mathcal{P}(n)$ – module

(G_n, α_n, β_n): morphism in $\mathcal{C}\mathrm{Mod}_{\mathcal{P}(n)}$

Towers of relative modules

$\mathcal{C}$ – small category

$i_n : \mathcal{P}(n) \hookrightarrow \mathcal{P}(n+1)$

$$\Big((\mathcal{D}_n, \Theta_n),\ \mathcal{C} \xrightarrow{T_n} \mathcal{D}_n,\ (T_n, \Theta_n) \xrightarrow{(G_n, \alpha_n, \beta_n)} (T_{n+1}, i_n^* \Theta_{n+1}) \Big)_{n \geq 0}$$

functor

$\mathcal{P}(n)$ – module

morphism in $\mathcal{C}\mathrm{Mod}_{\mathcal{P}(n)}$

$\Rightarrow$ category $\mathcal{C}\mathrm{Tow}_{\mathcal{P}(*)}$

Towers of relative modules induce towers of comonads

$\mathcal{C}$ - small cat, $\mathcal{M}$ - pted simplicial model cat

$$\Rightarrow \exists\, K^{\mathcal{C},\mathcal{M}} : {}^{\mathcal{C}}\mathrm{Tow}^{op}_{\mathcal{P}(*)} \longrightarrow \mathrm{Tow}\left(\mathrm{Comon}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}\right)$$

built at each level as in the case of ${}^{\mathcal{C}}\mathrm{Mod}_{\mathcal{P}(n)}$.

The tower pipeline $\mathcal{C}, \mathcal{M}$ as before

$$\mathcal{C}^{op}_{Tow_{\mathcal{P}(*)}} \times Fun(\mathcal{C}, \mathcal{M})$$

$$\mathbb{K}^{\mathcal{C},\mathcal{M}}_{\bullet} \times Id$$

$$\downarrow$$

$$Tow(Comon_{Fun(\mathcal{C},\mathcal{M})}) \times Fun(\mathcal{C}, \mathcal{M})$$

The tower pipeline $\mathcal{C}, \mathcal{M}$ as before

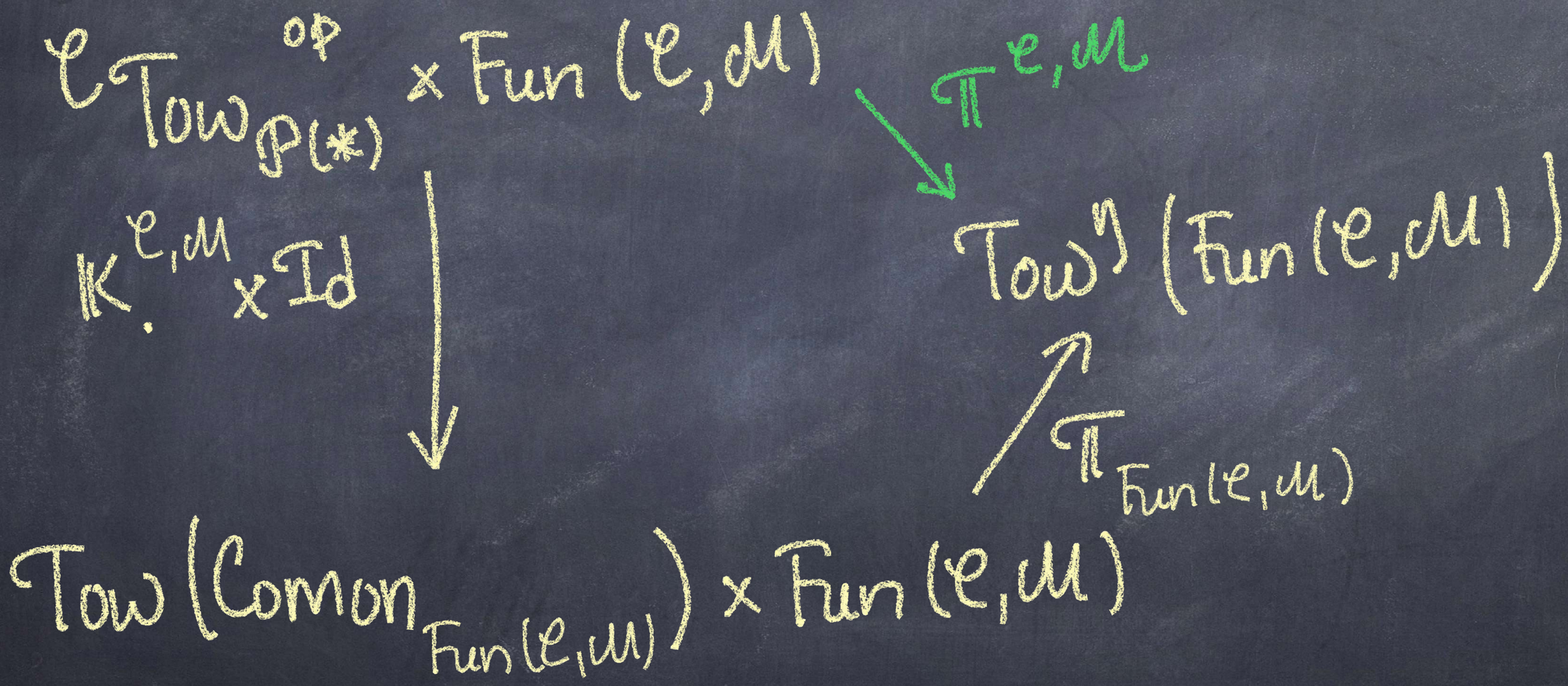
$$\mathcal{C}\mathrm{Tow}_{\mathcal{P}(*)}^{op} \times \mathrm{Fun}(\mathcal{C},\mathcal{M})$$

$$\mathbb{K}_{\cdot}^{\mathcal{C},\mathcal{M}} \times \mathrm{Id} \downarrow$$

$$\mathrm{Tow}(\mathrm{Comon}_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}) \times \mathrm{Fun}(\mathcal{C},\mathcal{M})$$

$$\mathrm{Tow}^{\eta}(\mathrm{Fun}(\mathcal{C},\mathcal{M})) \uparrow \pi_{\mathrm{Fun}(\mathcal{C},\mathcal{M})}$$

The tower pipeline $\mathcal{C}, \mathcal{M}$ as before



The calculus machine for functor categories

Precalculus

$\left((\mathcal{D}_n, \theta_n), \tau_n, (G_n, \alpha_n, \beta_n)\right) \in \mathrm{Ob}^{\mathcal{C}}\mathrm{Tow}_{\mathcal{P}(*)}$

is a precalculus if every $\tau_n : \mathcal{C} \to \mathcal{D}_n$ admits a left adjoint.

Precalculus

Notn: $\mathcal{D}_*$

$\left((\mathcal{D}_n, \theta_n), \tau_n, (G_n, \alpha_n, \beta_n)\right) \in \mathrm{Ob}^{\mathcal{C}}\mathrm{Tow}_{\mathcal{P}(*)}$

is a precalculus if every $\tau_n : \mathcal{C} \to \mathcal{D}_n$ admits a left adjoint.

From precalculus to calculus

If every obj. in $\mathcal{M}^{\Delta^{op}}$ is Reedy cofibrant and $\mathcal{D}_* \in {}^{\mathcal{C}}\mathrm{Tow}_{\mathcal{P}(*)}$ is a precalculus, then

$$\pi^{\mathcal{C},\mathcal{M}}(\mathcal{D}_*, -) : \mathrm{fun}(\mathcal{C},\mathcal{M}) \to \mathrm{Tow}^{\mathcal{Y}}(\mathrm{Fun}(\mathcal{C},\mathcal{M}))$$

is a calculus.

functors taking fibrant values

From precalculus to calculus

If every obj. in $\mathcal{M}^{\Delta^{op}}$ is Reedy cofibrant and $\mathcal{D}_* \in {}^{\mathcal{C}}\mathcal{T}ow_{\mathcal{P}(*)}$ is a precalculus, then

$$\pi^{\mathcal{C},\mathcal{M}}(\mathcal{D}_*, -) : fun(\mathcal{C},\mathcal{M}) \longrightarrow \mathcal{T}ow^{y}(Fun(\mathcal{C},\mathcal{M}))$$

is a calculus, which is a homotopy calculus if the following condition holds

The homotopy condition

$\forall S \in \mathcal{P}(n)$,

$$\mathcal{C} \xrightarrow{T_n} \mathcal{D} \xrightarrow{\Theta(-,S)} \mathcal{D} \xrightarrow{L_n} \mathcal{C}$$

preserves weak equivalences, $\forall n \geq 0$.

Homogeneity

- Stability + nice fibrant replacement $\Rightarrow$ $\Lambda_n \pi^{e,m}(\mathcal{D}_*, -)$ is indeed n-homogeneous

Perspectives

- Dual cocalculus theory for "telescopes of monads"
- ∞-categorical version
- Examples!


Thank you!