# Cauchy Completeness and Adjoints in Double Categories

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## Motivation

1. Lawvere (1973): Cauchy completeness by considering a (generalized) metric space as a category enriched in  $[0, \infty]$ .

2. Paré (2021): Cauchy completeness in double categories, and showed an (S, R)-modules M has a right adjoint in  $\mathbb{R}$ ing of commutative rings iff it is finitely generated and projective over S.

3. N./Wood (2017):  $-\otimes_S M$  on S-Mod has a left adjoint iff M is fg projective over S, for commutative rings, rigs, (and quantales).

Goals:

- More examples in double categories (Loc, Topos, Top, Quant).
- Remove commutativity from 3. and relate it directly to 2.

## **Double Categories**

A double category  $\mathbb D$  is a pseudo internal category in  $\operatorname{CAT}$ 

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\bullet} \mathbb{D}_1 \xrightarrow{s \\ \underbrace{\leftarrow \mathrm{id} \bullet}_{t} \\ t} \mathbb{D}_0$$

Objects X of  $\mathbb{D}_0$ , called objects of  $\mathbb{D}$ 

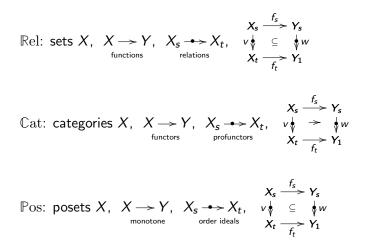
Morphisms  $X \xrightarrow{f} Y$  of  $\mathbb{D}_0$ , called horizontal morphisms of  $\mathbb{D}$ 

Objects  $X_s \xrightarrow{\nu} X_t$  of  $\mathbb{D}_1$ , called vertical morphism of  $\mathbb{D}$ 

 $\begin{array}{ll} X_s \xrightarrow{f_s} Y_s \\ \text{Morphisms} & v_{\psi}^{\dagger} & \varphi & \psi_w \text{ of } \mathbb{D}_1, \text{ called cells of } \mathbb{D} \\ & X_t \xrightarrow{f_t} Y_t \end{array}$ 

A cell is special if  $f_s$  and  $f_t$  are identity morphisms. The vertical morphisms and special cells form a bicategory denoted by  $Vert(\mathbb{D})$ .

### Examples



Met:  $\mathcal{V}$ -Cat for  $\mathcal{V} = [0, \infty]$ , Lawvere metric spaces

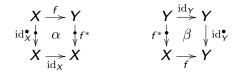
## Companions and Conjoints

A companion for  $X \xrightarrow{f} Y$  is a vertical morphism  $X \xrightarrow{f_*} Y$  and cells



whose horizontal and vertical compositions are identities.

A conjoint for f is a vertical morphism  $Y \xrightarrow{f^*} X$  and cells



Note: Rel, Cat, Pos, and Met have all companions and conjoints.

# Cauchy Completeness

### Proposition

If f has a companion and conjoint, then  $f_* \dashv f^*$  in  $Vert(\mathbb{D})$ .

### Definition

An object Y of  $\mathbb{D}$  is Cauchy complete if every left adjoint vertical morphism  $v: X \dashrightarrow Y$  is the companion of some  $f: X \longrightarrow Y$ .

#### Exercise

Every set Y is Cauchy complete in  $\mathbb{R}el$ .

#### Remark

Cauchy completeness was considered in the 70s and 80s for metric spaces, categories, and posets (see Borceux/Dejean).

## Locales

Companions and conjoints played a role in a double category construction [N 2012] of exponentials of locally closed inclusions for locales, toposes, and topological spaces using Artin-Wraith glueing.

$$\mathbb{L}\text{oc: locales } X, \quad X \xrightarrow{f} Y, \quad X_s \xrightarrow{f} X_t, \quad v \oint \geq \psi w$$

$$\lim_{\text{locale maps}} Y, \quad X_s \xrightarrow{f} X_t, \quad v \oint \geq \psi w$$

$$X_t \xrightarrow{f_t} Y_t$$

Note: A "locale map" f has a finite  $\wedge$ -preserving left adjoint  $f^*$ .

### Proposition

Every locale is Cauchy complete in Loc.

### Proof.

Suppose  $v: X \dashrightarrow Y$  is left adjoint to  $w: Y \dashrightarrow X$  in Loc. Then  $vw \ge id_Y^{\bullet}$  and  $id_X^{\bullet} \ge wv$ , and so  $w \dashv v$  as poset maps. Since v preserves finite meets, it follows that v is a locale morphism such that  $v_* = v$  in Loc.

### Toposes

Note: A "geometric morphism" f has a left exact left adjoint  $f^*$ .

### Proposition

Every topos is Cauchy complete in Topos.

### Proof.

Suppose  $v \dashv w$  in Topos. Then we have cells  $vw \twoheadleftarrow id_Y^{\bullet}$  and  $id_X^{\bullet} \twoheadleftarrow wv$ , satisfying the adjunction identities, and so  $w \dashv v$  as functors. Since v preserves finite limits, it follows that v is a geometric morphism such that  $v_* = v$  in Topos.

# **Topological Spaces**

$$\mathbb{T} \text{op: top spaces } X, \quad X \longrightarrow Y, \quad \underbrace{X_s \longrightarrow X_t}_{\text{cont maps}}, \quad \underbrace{X_s \longrightarrow X_t}_{\text{lex}}, \quad \begin{array}{c} \mathcal{O}(X_s) \xrightarrow{\mathcal{O}(f_s)} \mathcal{O}(Y_s) \\ \psi \downarrow \quad \supseteq \quad \psi w \\ \mathcal{O}(X_t) \xrightarrow{\mathcal{O}(f_t)} \mathcal{O}(Y_t) \end{array}$$

Recall [PTJ] a space Y is sober iff morphisms  $f: \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$  of locales correspond bijectively to continuous maps  $f: X \rightarrow Y$ .

### Proposition

A space Y is Cauchy complete in Top iff it is a sober space.

### Proof.

Left adjoints  $X \dashrightarrow Y$  in  $\mathbb{T}$ op are the left adjoints  $\mathcal{O}(X) \dashrightarrow \mathcal{O}(Y)$ in  $\mathbb{L}$ oc, and so Y is Cauchy complete in  $\mathbb{T}$ op iff it is sober.  $\Box$ 

# Quantales

Quant: quantales X, 
$$X \xrightarrow{f} Y$$
,  $X_s \xrightarrow{v} X_t$ ,  $v \notin A_t$ .  
Note: v is monotone with  $v(x)v(x') \le v(xx')$  and  $e \le v(e)$ .

. . . .

### Proposition

Every quantale Y is Cauchy complete in  $\mathbb{Q}$ uant.

#### Proof.

Suppose  $v \dashv w$  in Quant, where  $X \stackrel{v}{\dashrightarrow} Y$ . Since v is lax and preserves  $\bigvee$ , to see it is a quantale morphism, it suffices to show  $v(e_X) \leq e_Y$  and  $v(xx') \leq v(x)v(x')$ . But,  $e_X \leq w(e_Y)$  and

$$xx' \leq wv(x)wv(x') \leq w(v(x)v(x'))$$

Thus, Y is Cauchy complete in  $\mathbb{Q}$ uant.

# Adjoints in Double Categories

Suppose  ${\cal V}$  is a bicomplete symmetric monoidal closed category, and consider the double category of monoids in  ${\cal V}$ 

$$\mathbb{Bim}(\mathcal{V}): \text{ monoids } R \text{ , } R \xrightarrow{f} S, R_s \xrightarrow{M} R_t, M_{\clubsuit} \xrightarrow{R_s} S_s$$
$$\underset{(R_t, R_s) \text{-bimods}}{\overset{(R_t, R_s) \text{-$$

Given an (S, R)-module M, there is a functor

$$M\otimes_R-:(R,Q)\operatorname{\!-Mod}\nolimits\longrightarrow(S,Q)\operatorname{\!-Mod}\nolimits$$

which has a right adjoint

$$SMod(M, -): (S, Q)-Mod \longrightarrow (R, Q)-Mod$$

For commutative rings,  $M \otimes_R -$  has a left adjoint iff M is fg projective as an R-module. Can we relate this to right adjoints to  $M: R \rightarrow S$ ? What about rigs/quantales? Non-commutative case?

# Adjoints in Double Categories

Theorem TFAE for  $M: R \rightarrow S$  with S-presentation  $\sqcup_{\alpha} S \rightrightarrows \sqcup_{\beta} S \rightarrow M$ . (a)  $M: R \rightarrow S$  has a right adjoint in  $\mathbb{B}im(\mathcal{V})$ . (b) (Q, S)-Mod $(\mathcal{V}) \xrightarrow{-\otimes_S M} (Q, R)$ -Mod $(\mathcal{V})$  has a left adjoint,  $\forall Q$ . (c) (Q, S)-Mod $(\mathcal{V}) \xrightarrow{-\otimes_S M} (Q, R)$ -Mod $(\mathcal{V})$  preserves limits. (d)  $SMod(M, S) \otimes_{S} M \xrightarrow{\theta} SMod(M, M)$  is invertible. Note:  $(b) \Rightarrow (c) \Rightarrow (d)$  is like [NW]; can prove  $(d) \Rightarrow (a) \Rightarrow (b)$ . Corollary TFAE for an (S, R)-module M over quantales (resp., rings, rigs).

(a) 
$$M: R \rightarrow S$$
 has a right adjoint in  $\mathbb{B}$ im.

(b)  $-\otimes_S M$  has a left adjoint.

(c) M is (resp., fg) projective as an S-module.

Note: One can prove (c) iff  $\theta$  is an invertible.

### References

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