(Co)algebraic analysis of social systems: from graphs to hypergraphs

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Outline of talk

I. Structural social science

II. Algebraic analysis of graphs: role and positional analysis

III. Higher-order relations

IV. Coalgebraic analysis of hypergraphs

I. Structural social science

What is structural social science?

For the last thirty years, empirical social research has been dominated by the sample survey. But as usually practiced, using random sampling of individuals, the survey is a sociological meatgrinder, tearing the individual from his social context and guaranteeing that nobody in the study interacts with anyone else in it. It is a little like a biologist putting his experimental animals through a hamburger machine and looking at every hundredth cell through a microscope; anatomy and physiology get lost, structure and function disappear, and one is left with cell biology....If our aim is to understand people's behavior rather than simply to record it, we want to know about primary groups, neighborhoods, organizations, social circles, and communities; about interaction, communication, role expectations, and social control.

Allen Barton, 1968

from The Development of Social Network Analysis, Freeman, 2004

What is structural social science?

Mainstream social research was and *still is* focused solely on studying individuals.

It neglects the *social* aspect of behaviour: how individuals interact and influence each other.

Structural social science: social science research that focuses on studying the relationships between individuals rather than the individuals themselves.

Social network analysis

Since 1960s/70s: development of formal methods centered on using networks to study relationships and patterns

The structural approach has an impact well beyond social science:

Most methods used today in Network Science were first developed by social network scientists.

II. Algebraic analysis of graphs: role and positional analysis

Multirelational graphs

Multirelational graphs:

- nodes: represent social actors or positions

  e.g., employees in a firm, members of a family, alliance groups within a political party

- Edges (directed or not, labelled by a relation): represent relationships between social actors or positions

  e.g., “being the supervisor of”, “being in the same generation as”, “having cosponsored the same bill”

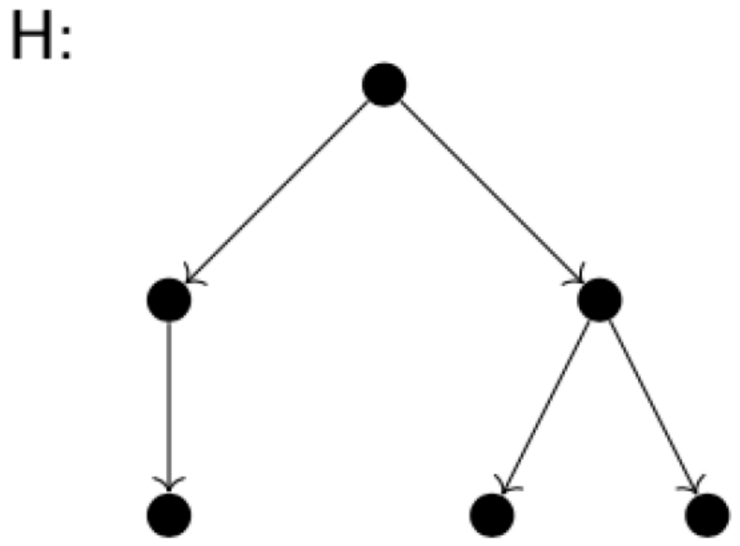
Multirelational graphs

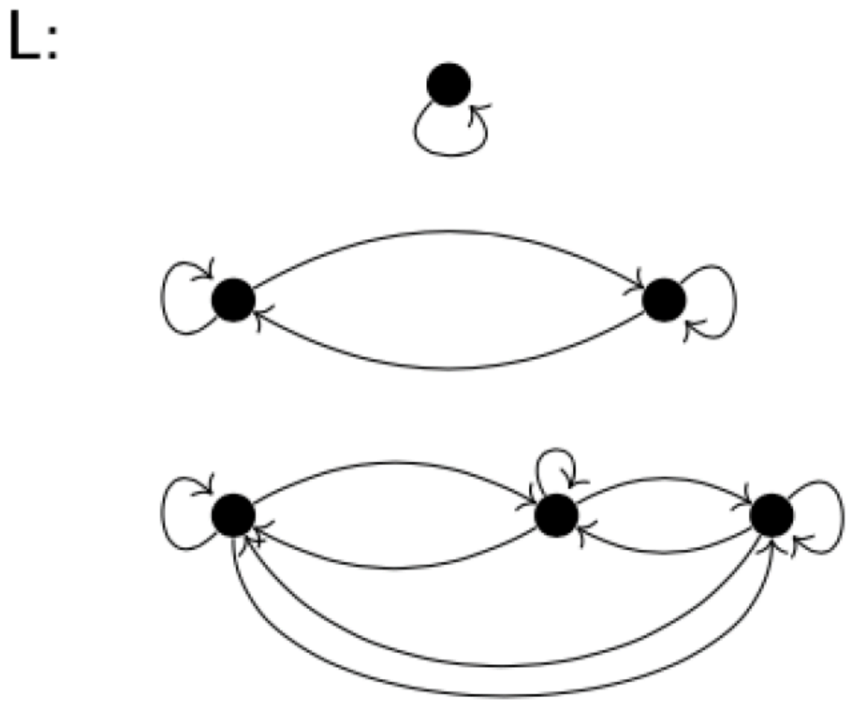
More formally, we define a ***k*-relational graph** to be a pair $\left(V, \left\{A_i\right\}_{i=1}^{r}\right)$ where:

- V is a finite set of **vertices**
- $A_1, \ldots, A_r$ are $|V| \times |V|$-matrices with Boolean entries (i.e., 0 or 1) that we call ma**trices of relations**

(Running) example

Here, nodes represent employees in a firm, the relation H encodes “being the supervisor of”, and relation L encodes “being able to sub in for”



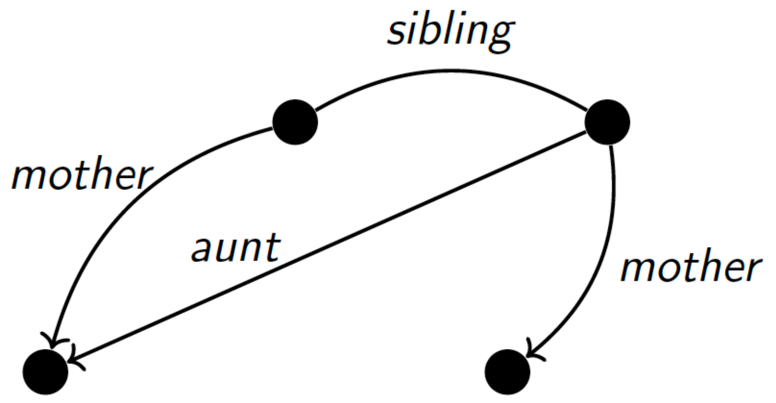


Position and roles

Social positions: collections of actors who are similar in their relationships to others

Social roles: patterns of relationships among actors or positions

Example of social role: In a kinship network we have that the relations “sibling of a mother” and “aunt" tie the same pairs of actors together, and thus correspond to a “role”.



Positional analysis on multirelational graphs

Goal: study **blockmodels** of a given multirelational graph, i.e.,:

multirelational graph $\twoheadrightarrow$ reduced multirelational graph

surjective graph homomorphism

nodes are positions

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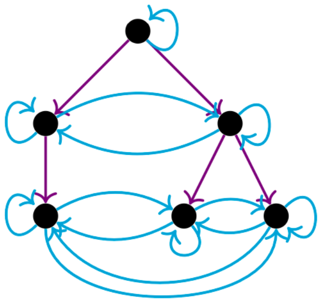
Example:

$\xrightarrow{H} = \longrightarrow$

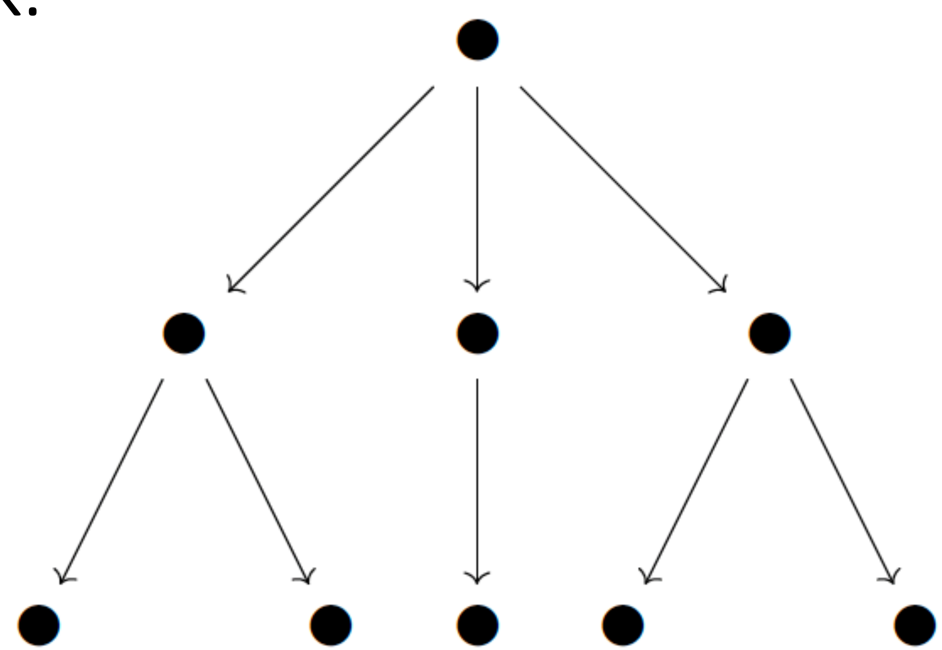
$\xrightarrow{L} = \longrightarrow$



$\twoheadrightarrow$

Equivalence relations that capture similarity

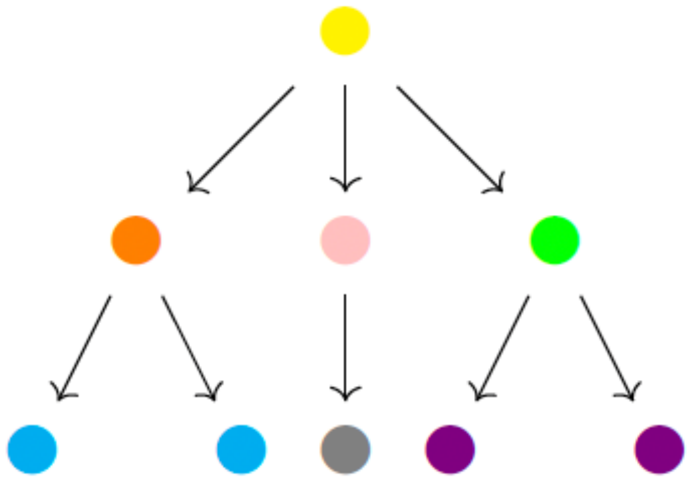
Given this network:



We want to partition the nodes into blocks of similar nodes.

Structural equivalence

Structural equivalence: two nodes are structurally equivalent if they have exactly the same neighbours



Interpretation: two actors are structurally equivalent if they supervise the same people and are supervised by exactly the same.

Structural equivalence

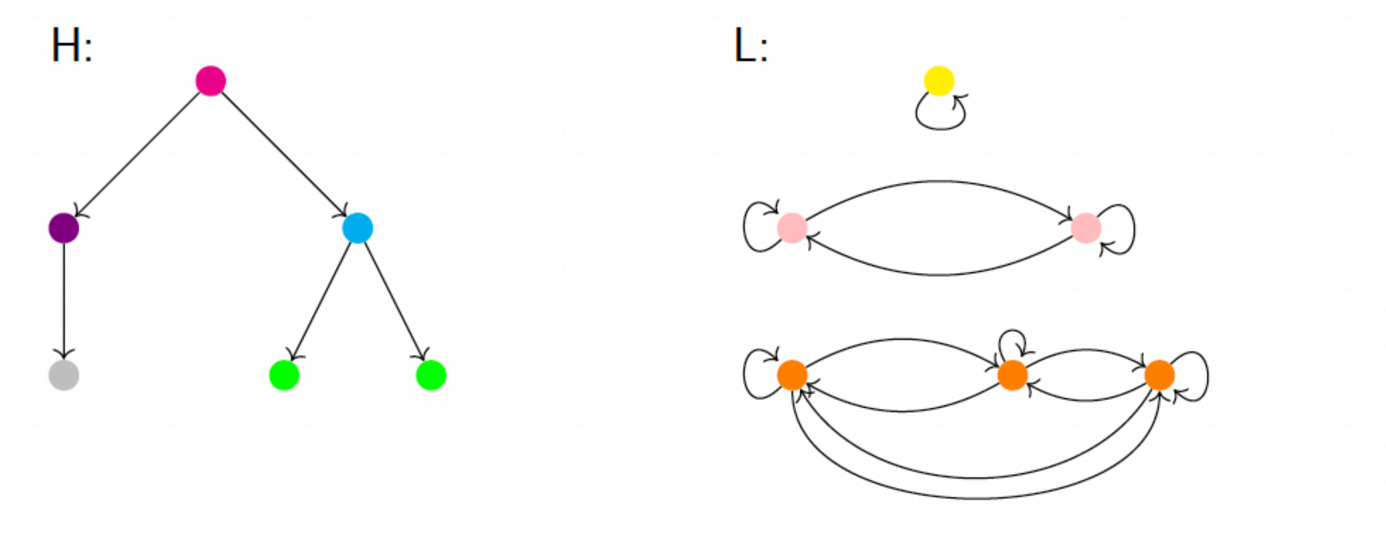
Let $\left(V, \{A_i\}_{i=1}^{r}\right)$ be a multirelational graph.

An equivalence relation $R \subseteq V \times V$ is a **structural equivalence** iff for all $v, v' \in V$ we have:

$(v, v') \in R$ iff v and v' have exactly the same neighbours under all relations.

Structural equivalence

Considering each relation separately, we obtain the following partitions for structural equivalence:



However, no two pairs of nodes in the network have exactly the same neighbours under *both* relations.

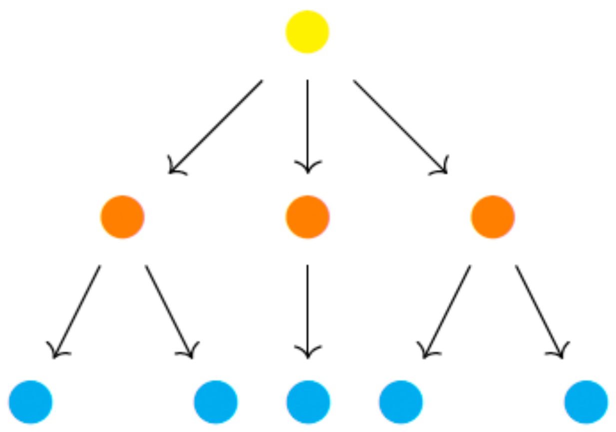
Regular equivalence

Problem with structural equivalence: for two actors to occupy the same social position we require them to interact with the same people.

Regular equivalence

Problem with structural equivalence: for two actors to occupy the same social position we require them to interact with the same people.

Regular equivalence: two nodes are regularly equivalent if they have neighbours who themselves are regularly equivalent.



Regular equivalence

Let $\left(V, \{A_i\}_{i=1}^{r}\right)$ be a multirelational graph.

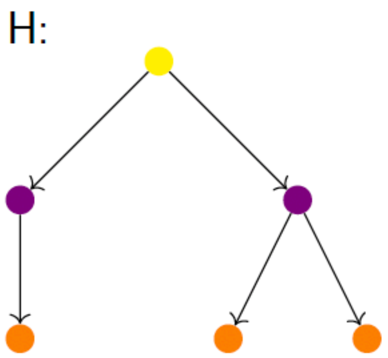
An equivalence relation $R \subseteq V \times V$ is a **regular equivalence** iff for all $v, v' \in V$ we have $(v, v') \in R$ implies, for all $i = 1, \ldots, r$:

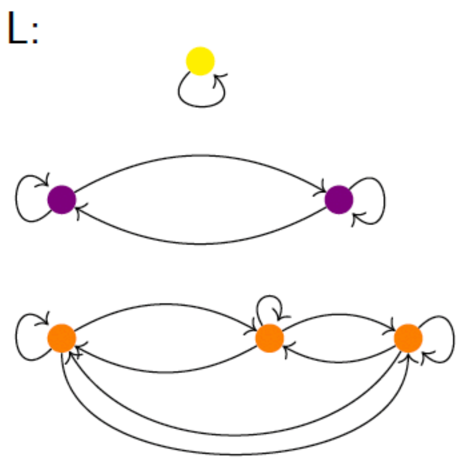
- if $v \xrightarrow{i} w$ is an edge, then there exists w' with $(w, w') \in R$ such that $v' \xrightarrow{i} w'$ is an edge
- if $v \xleftarrow{i} w$ is an edge, then there exists w' with $(w, w') \in R$ such that $v' \xleftarrow{i} w'$ is an edge.

Regular equivalence

Note: The set of regular equivalence relations on a given set of vertices is a complete lattice with respect to inclusion.

The maximum regular equivalence for our running example, under *both* relations, is:





Positional analysis on multirelational graphs

multirelational graph $\twoheadrightarrow$ reduced multirelational graph

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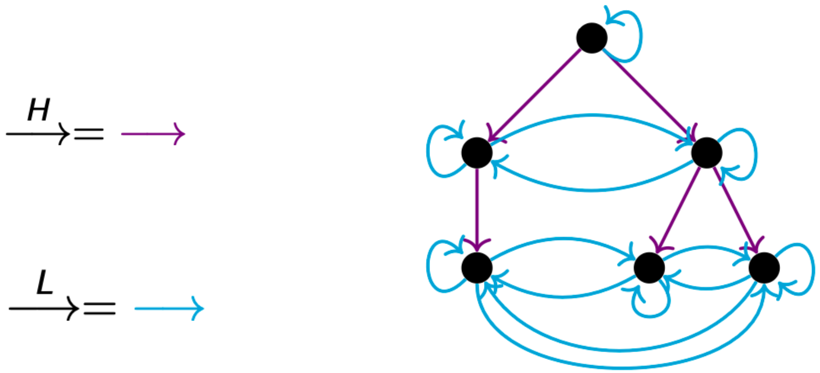
Example:



$\twoheadrightarrow$

Role analysis

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Role analysis's aim is to study all possible compound relations and to find or impose equalities between them.

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Definition. Given a multirelational graph $\left(V, \{A_i\}_{i=1}^{r}\right)$ its **semigroup of roles** is the semigroup $SG(A_1, \ldots, A_r)$ generated by $A_1, \ldots, A_r$, with binary operation given by Boolean matrix multiplication.

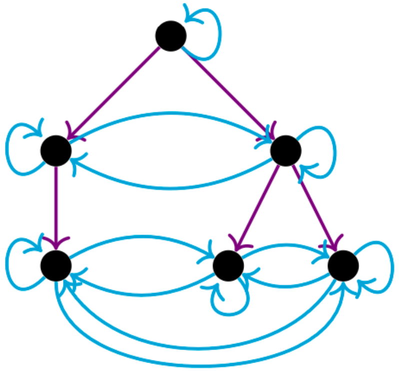
Goal: study **homomorphic reductions** of the semigroup of roles, e.g.,

$$SG(A_1, \ldots, A_r) \twoheadrightarrow S$$

Running example

$\xrightarrow{H} = \color{purple}{\longrightarrow}$

$\xrightarrow{L} = \color{cyan}{\longrightarrow}$



$\circ$	H	L	HL	HH
H	HH	HL	HH	0
L	HL	L	HL	HH
HL	HH	HL	HH	0
HH	0	HH	0	0

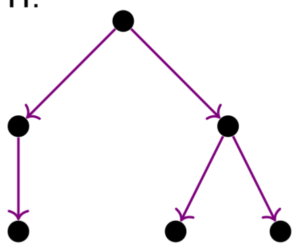
$\twoheaddownarrow$

$\circ$	H	id	HH
H	HH	H	0
id	H	id	HH
HH	0	HH	0

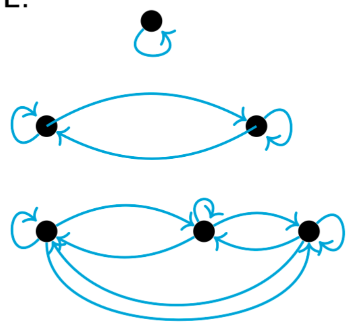
Quotient by congruence relation obtained by setting L=id

Combining positional and role analysis

H:



L:



$$A_H = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$\circ$	H	L	HL	HH
H	HH	HL	HH	0
L	HL	L	HL	HH
HL	HH	HL	HH	0
HH	0	HH	0	0

$$\Downarrow$$

$$\overline{A}_H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overline{A}_L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\circ$	H	id	HH
H	HH	H	0
id	H	id	HH
HH	0	HH	0

Question: Given $\left(V, \{A_i\}_{i=1}^{r}\right) \twoheadrightarrow \left(\bar{V}, \{\bar{A}_i\}_{i=1}^{r}\right)$ under what conditions is $SG(\bar{A}_1, \ldots, \bar{A}_r)$ a homomorphic reduction of $SG(A_1, \ldots, A_r)$?

$$
\begin{tikzcd}
G = \left(V, \{A_i\}_{i=1}^{k}\right) \arrow[rrr, mapsto] \arrow[d, two heads, "\phi"'] & & & \mathrm{SG}\left(\{A_i\}_{i=1}^{k}\right) \arrow[d, two heads, dashed, "\mathrm{SG}(\phi)?"] \\
G' = \left(\overline{V}, \{\overline{A_i}\}_{i=1}^{k}\right) \arrow[rrr, mapsto] & & & \mathrm{SG}\left(\{\overline{A_i}\}_{i=1}^{k}\right)
\end{tikzcd}
$$

Functoriality of semigroup of roles assignment

$\mathrm{Graph}_{\mathrm{surj}}$

- objects: k-relational graphs
- morphisms: surjective k-relational graph homomorphisms that are locally surjective

$\mathrm{SemiGroup}_{\mathrm{surj}}$

- objects: semigroups
- morphisms: surjective semigroup homomorphisms

Theorem (O., Porter 2020 & Motamed, O., Roff 2025)

The semigroup of roles assignment induces a functor

$$\mathrm{Role}\colon \mathrm{Graph}_{\mathrm{surj}} \to \mathrm{SemiGroup}_{\mathrm{surj}} .$$

Functoriality of semigroup of roles assignment

$\mathrm{Graph}_{\mathrm{surj}}$

- objects: k-relational graphs
- morphisms: surjective k-relational graph homomorphisms that are locally surjective

These give exactly the blockmodels associated to regular equivalences

$\mathrm{SemiGroup}_{\mathrm{surj}}$

- objects: semigroups
- morphisms: surjective semigroup homomorphisms

Theorem (O., Porter 2020 & Motamed, O., Roff 2025)

The semigroup of roles assignment induces a functor

$$\mathrm{Role}\colon\ \mathrm{Graph}_{\mathrm{surj}} \to \mathrm{SemiGroup}_{\mathrm{surj}}\ .$$

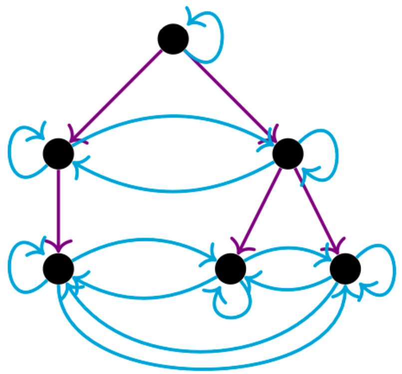
What is this good for?

- Gives a natural choice for homomorphic reductions in role analysis
- Allows to tie several iterations of positional analysis to role analysis
- Conceptual clarity
- Stability questions: we could ask, e.g., is Role Lipschitz?

Functoriality theorem in action

$\xrightarrow{H} = \longrightarrow$

$\xrightarrow{L} = \longrightarrow$



$\longmapsto$

∘	H	L	HL	HH
H	HH	HL	HH	0
L	HL	L	HL	HH
HL	HH	HL	HH	0
HH	0	HH	0	0

$\twoheaddownarrow$

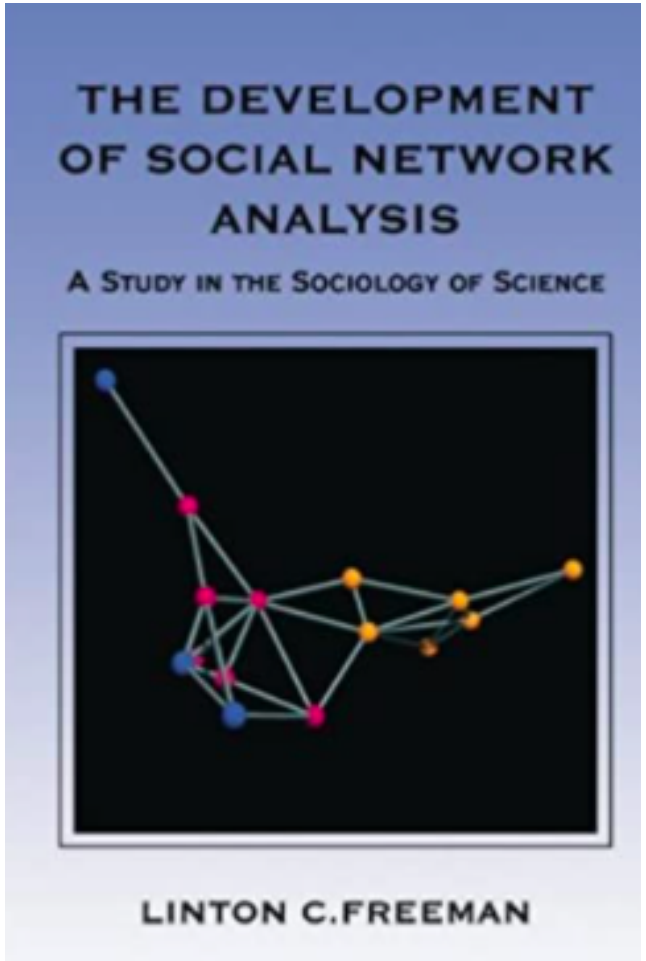
$\twoheaddownarrow$

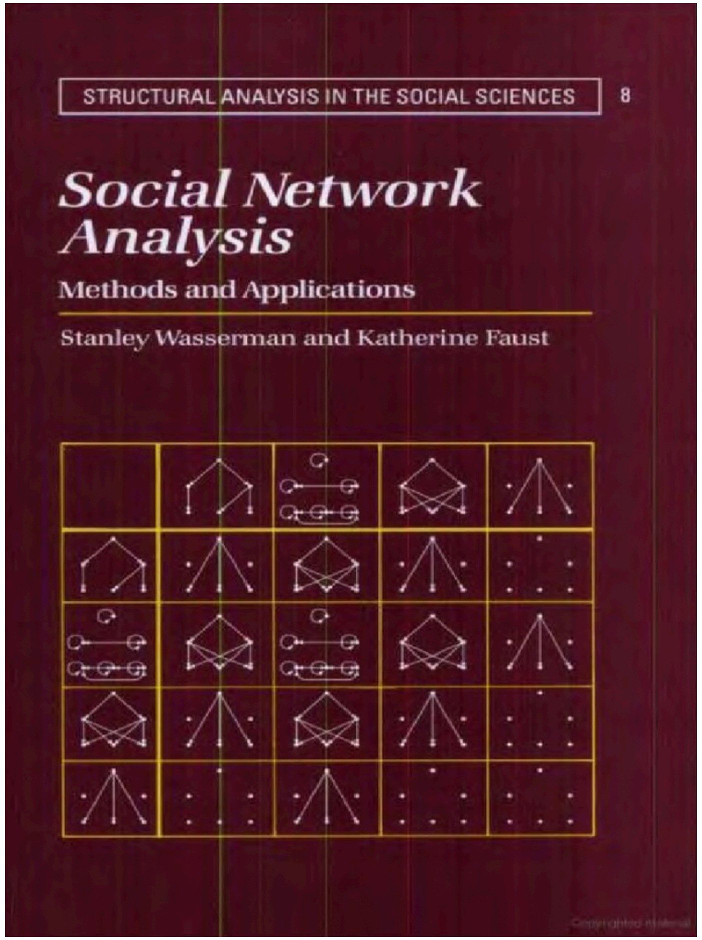
$\longmapsto$

$\circ$	$\overline{H}$	$\overline{L}$	$\overline{H}\,\overline{H}$
$\overline{H}$	$\overline{H}\,\overline{H}$	$\overline{H}$	0
$\overline{L}$	$\overline{H}\,\overline{H}$	$\overline{L}$	$\overline{H}\,\overline{H}$
$\overline{H}\,\overline{H}$	0	$\overline{H}\,\overline{H}$	0

Here we are barely scratching the surface

Some further reading..





III. Higher-order relations

Why higher-order models?

Example: coauthorship systems.

We consider papers in different disciplines, e.g.

- algebraic topology (A)
- deep learning (D)

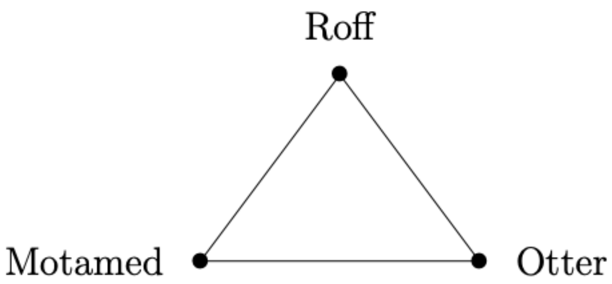
Papers in either A or D encode co-authorship relations between authors of each paper.

How shall we model this?

Why higher-order models?

- Undirected graphs, i.e. (V, E) with $E \subseteq \mathscr{P}_2(V)$

We encode pairwise relationships between authors, but we loose more refined information. We label edges according to the topics.



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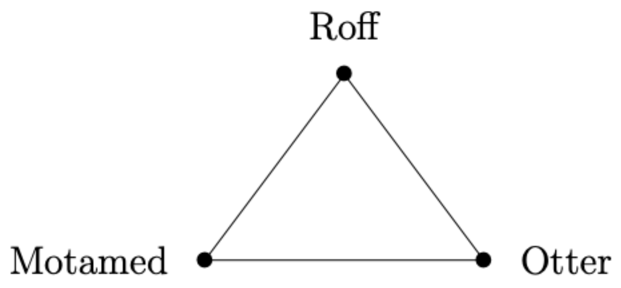
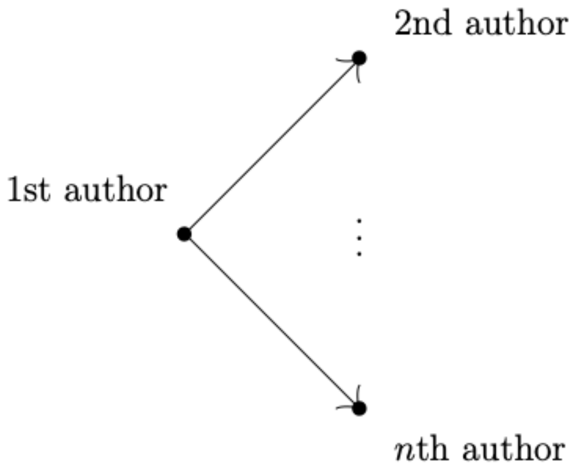


- Directed graphs, i.e., (V, E) with $E \subseteq V \times V$

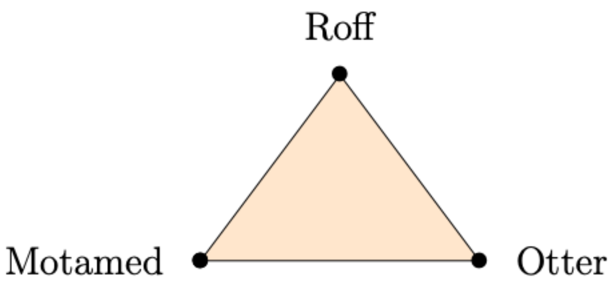
We can encode, e.g., first or last authorship in applied papers. We label edges according to the topics.



Why higher-order models?

- Simplicial complexes, i.e., (V, Σ) with $\Sigma \subseteq \mathscr{P}(V)$ downward closed

We encode a paper in topic A or D with n authors by an $n-1$-simplex labelled by A or D.



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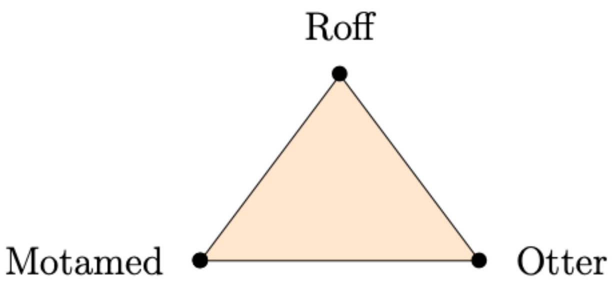


- Simplicial complexes, i.e., (V, Σ) with $\Sigma \subseteq \mathscr{P}(V)$ downward closed

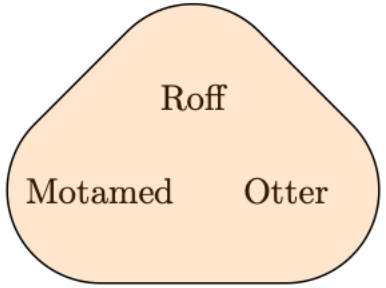
We encode a paper in topic A or D with n authors by an n-simplex labelled by A or D.

Note: here we also encode information about all co-authorship between subsets of the set of n authors.
We can interpret this model as follows: simplices encode co-authorship between their vertices, and maximal simplices represent papers.

Why higher-order models?

- Undirected hypergraphs, i.e., (V, H) with $\mathrm{H} \subseteq \mathscr{P}(V)$

Hyperedges represent papers.



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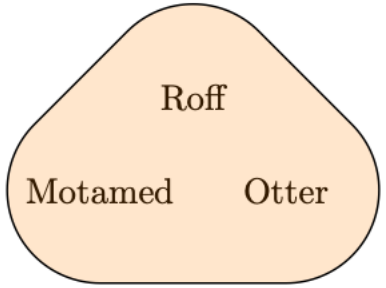
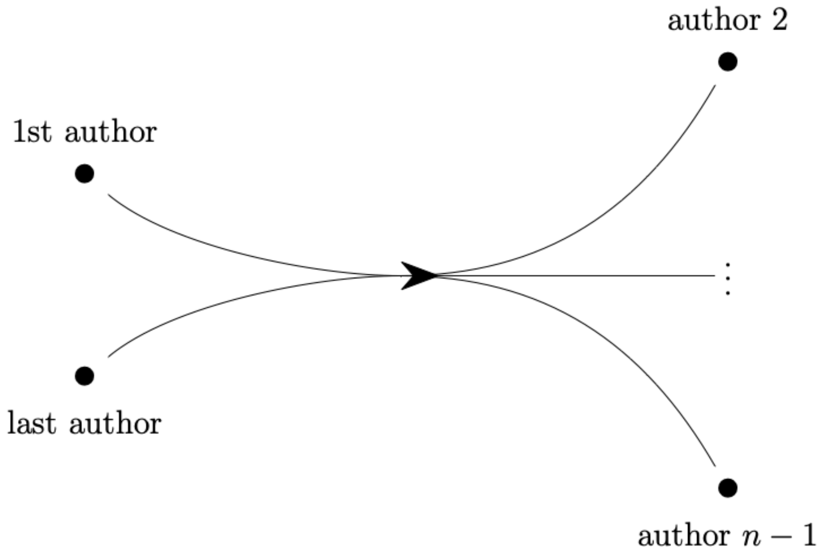


- Directed hypergraphs, i.e., (V, D) with $D \subseteq \mathscr{P}(V) \times \mathscr{P}(V)$

Directed hyperedges represent papers.
We can additionally, e.g., encode information about first and last author.



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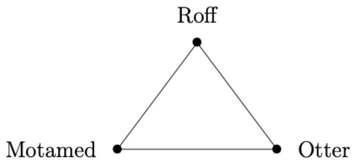
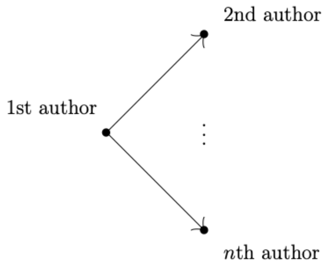
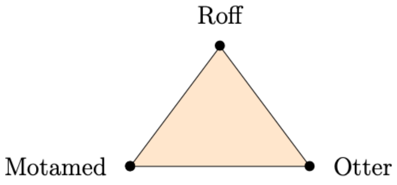
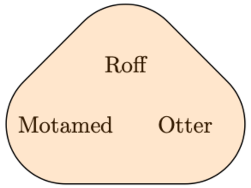
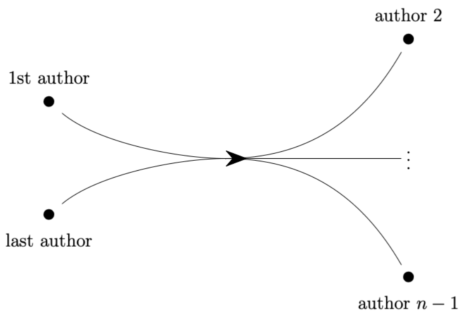










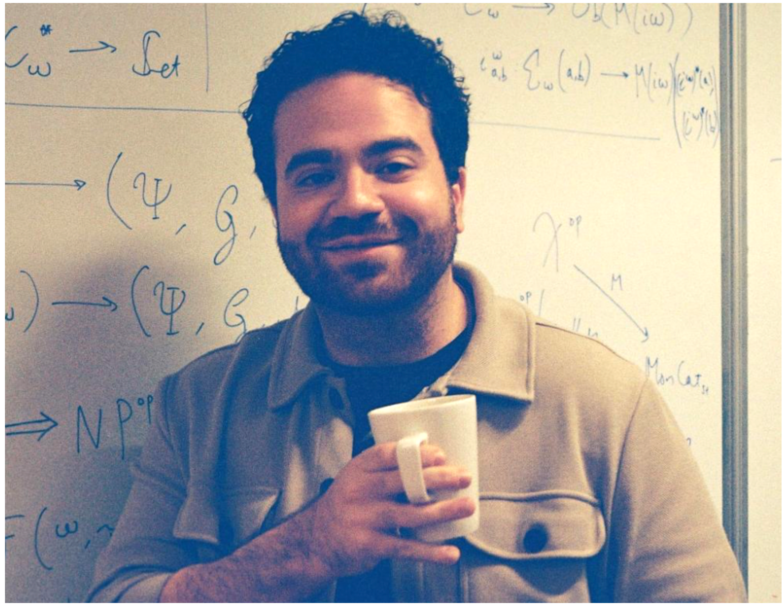
Each model captures different type of information: in general no model is better than any other.

The objective

Generalise, to hypergraphs/simplicial complexes:

- Positional analysis
- Role analysis
- Tie the analyses together through a functoriality result

AMS MRC 2022 on
Applied Category Theory



Nima Motamed, Utrecht University

Nima's insight: regular equivalences can be generalised through bisimulations on coalgebras

IV. Coalgebraic analysis of hypergraphs

Coalgebras

Categorical approach to systems theory: coalgebras arose in the 1970s as a way to develop a unified framework for sequential machines and control systems.

In a nutshell: coalgebras capture the behaviour of a system.

$$\text{states} \longrightarrow \text{transitions}$$

transition assigning map

Example: non-deterministic transition systems

Let X be a finite set.

A **non-deterministic transition system** is a map $\alpha \colon X \to \mathscr{P}(X)$.

Example: non-deterministic transition systems

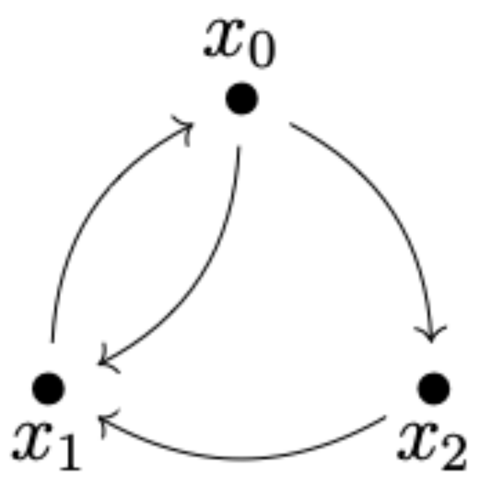
Let X be a finite set.

A **non-deterministic transition system** is a map $\alpha : X \to \mathscr{P}(X)$.

Example: $X = \{x_0, x_1, x_2\}$

$$\alpha(x_0) = \{x_1, x_2\}$$
$$\alpha(x_1) = \{x_0\}$$
$$\alpha(x_2) = \{x_1\}$$

Such a system can be represented graphically by drawing a directed edge from each state to each of the states in its set of transitions:



Example: labelled non-deterministic transition systems

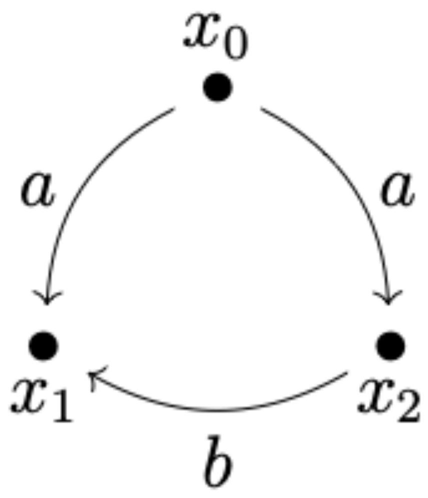
Let X and A be finite sets.

A **labelled non-deterministic transition systems** is a map $\alpha\colon X \to \mathscr{P}(A \times X)$.

Example: $X = \left\{x_0, x_1, x_2\right\}$ and $A = \{a, b\}$.

$$\alpha\left(x_0\right) = \left\{(a, x_1), (a, x_2)\right\}$$

$$\alpha\left(x_1\right) = \varnothing$$
$$\alpha\left(x_2\right) = \left\{(b, x_1)\right\}$$



Example: discrete probabilistic systems: Markov chains

Let X be a finite set, and let $\mathscr{D}(X)$ denote the set of discrete probability distributions on X:

$$\mathcal{D}(X) = \left\{ \mu \colon X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1 \right\} .$$

A **discrete probabilistic system** is a map $\alpha \colon X \to \mathscr{D}(X)$.

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Example: $X = \{x_0, x_1, x_2\}$

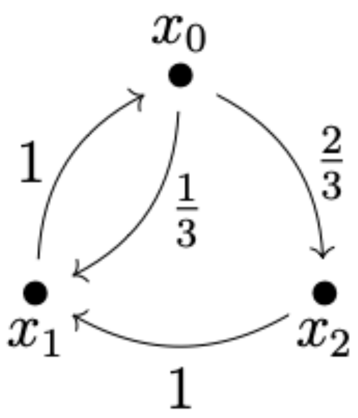
$\alpha(x_1) = \delta_{x_0}$

$\alpha(x_2) = \delta_{x_1}$

$\alpha(x_2) = \mu$

where μ is defined as:

$$\begin{aligned} \mu \colon x_0 &\mapsto 0 \\ x_1 &\mapsto \frac{1}{3} \\ x_2 &\mapsto \frac{2}{3} . \end{aligned}$$



Coalgebra: definition

Let $T: Set \to Set$ a functor. A **T-coalgebra** is a pair (A, α) where A is a set and $\alpha: A \to TA$ is a map of sets.

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Example: graphs as coalgebras. Given a directed graph (V, E), we can encode it by specifying its *out-neighborhood function*

$$\begin{aligned} N_G^{\text{out}} : V &\to \mathcal{P}(V) \\ v &\mapsto \{w \in V \mid (v, w) \in E\} \end{aligned}$$

We have:

Let $C = Set$. The $\mathscr{P}$-coalgebras are directed graphs.

Multi-relational graphs as coalgebras

More generally, k-relational directed graphs are F-coalgebras where

$$F: Set \to Set$$

is the functor induced by the object-level map $V \longmapsto \mathscr{P}(A \times V)$

where $A = \{1,\ldots, k\}$.

T-coalgebra homomorphisms

Let $T\colon Set \to Set$ a functor.

A **T-coalgebra homomorphism** $(A, \alpha) \to (B, \beta)$ is a map $f\colon A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

Example: homomorphisms of $\mathscr{P}$-coalgebras

Given two directed graphs $G = (V, E)$ and $G' = (V', E')$, a coalgebra homomorphism $\left(V, N_G^{out}\right) \to (V', N_{G'}^{out})$ is a map $V \to V'$ such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "N_G^{out}"'] & V' \arrow[d, "N_{G'}^{out}"] \\
\mathcal{P}(V) \arrow[r, "\mathcal{P}(f)"'] & \mathcal{P}(V')
\end{tikzcd}$$

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This is equivalent to the following two conditions:

(i) Edges in G are sent to edges in G'

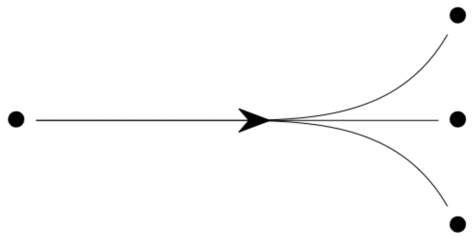
"f is a graph homomorphism"

(ii) $\forall v \in V$ and for any edge $f(v) \to y$ in G' there exists $v' \in V$ such that $v \to v'$ is an edge in G.

"f reflects edges"

Hypergraphs as $\mathscr{P}\mathscr{P}$-coalgebras

We restrict ourselves to a special type of directed hypergraph, in which the tail of each directed hyperedge is a singleton:



Note: everything can be extended to more general hypergraphs if one considers *di*algebras instead of coalgebras.

In what follows: A (directed) **hypergraph** is a pair $\mathrm{H} = (V, R)$ where V is a finite set and R $\subseteq V \times \mathscr{P}(V)$.

Hypergraphs as $\mathscr{P}\mathscr{P}$-coalgebras

Every hypergraph can be specified by its *hyper-neighbourhood* function:

$$\begin{aligned} V &\to \mathcal{P}(\mathcal{P}(V)) \\ v &\mapsto \{X \subseteq V \mid (v, X) \in R\} \end{aligned}$$

Let $C = Set$. The $\mathscr{P}\mathscr{P}$-coalgebras are directed hypergraphs.

Multirelational hypergraphs as coalgebras

An ***r*-relational hypergraph** is a pair $H = \left(V, \{R_i\}_{i=1, \ \ldots, \ r}\right)$ such that each $\left(V, R_i\right)$ is a hypergraph.

k-relational hypergraphs are F-coalgebras where

$F: Set \rightarrow Set$

is the functor induced by the object-level map

$V \longmapsto \mathscr{P}(A \times \mathscr{P}(V))$

where $A = \{1, \ldots, \ k\}$.

Bisimulations

Intuition: allow to study a directed graph by focusing only on the movements that are possible along its edges.

Introduced independently in the 1970s in computer science, modal logic and set theory.

"The independent discovery of bisimulation in three different fields suggests that only limited exchanges and contacts among researchers existed at the time." Sangiorgi, 2009

Sangiorgi, D. 2009. On the origins of bisimulation and coinduction. *ACM Trans. Program. Lang. Syst.*

Bisimulation equivalence

Intuitively, a bisimulation equivalence on a transition system (A, α) is an equivalence relation $R \subseteq A \times A$ that respects the transition structure.

Given a functor $T\colon Set \to Set$ and a T-coalgebra (A, α) a **T-bisimulation equivalence** on (A, α) is an equivalence relation $R \subseteq A \times A$ together with a T-coalgebra structure $\rho\colon R \to T(R)$ such that the projection- induced maps $A \overset{p_1}{\leftarrow} R \overset{p_2}{\rightarrow} A$ are T -coalgebra homomorphisms, or in other words, such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "\alpha"'] & R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\rho"] & A \arrow[d, "\beta"] \\
TA & TR \arrow[l, "Tp_1"] \arrow[r, "Tp_2"'] & TA
\end{tikzcd}$$

$\mathscr{P}$-bisimulation equivalences are regular equivalences

Proposition: $\mathscr{P}$-bisimulation equivalences on $\mathscr{P}$-coalgebras are exactly regular equivalences on directed graphs.

A similar result holds for the multirelational setting.

$\mathscr{P}$-bisimulation equivalences are regular equivalences

Proposition: $\mathscr{P}$-bisimulation equivalences on $\mathscr{P}$-coalgebras are exactly regular equivalences on directed graphs.

A similar result holds for the multirelational setting.

This gives us a way to extend regular equivalences to hypergraphs: these are given by $\mathscr{P}\mathscr{P}$-bisimulation equivalences on $\mathscr{P}\mathscr{P}$-coalgebras.

Positional analysis for T-coalgebras

Thus, we can generalise the notion of blockmodel (coming from a regular equivalence) of a multirelational graph to, in particular, multirelational hypergraphs:

Definition. Let $T\colon Set \to Set$ and $\alpha\colon A \to TA$ a coalgebra and $\rho\colon R \to TR$ a bisimulation equivalence on (A, α). The **T-blockmodel of (A, α) with respect to ρ** is the coequalizer in coalg(T) of

$$(R, \rho) \underset{p_2}{\overset{p_1}{\rightrightarrows}} (A, \alpha)$$

..but wait, there is more!

The powerset functor also carries the structure of a *monad, i.e., there are natural transformations*

$$\eta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{P} \text{ and } \mu : \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}$$

satisfying unitality and associativity properties, where

$$\begin{aligned} \eta_X \colon X &\to \mathscr{P}(X) \\ x &\mapsto \{x\} \end{aligned}$$

and

$$\begin{aligned} \mu_X : \mathcal{P}\mathcal{P}(X) &\to \mathcal{P}(X) \\ U &\mapsto \bigcup_{V \in U} V. \end{aligned}$$

Semigroup of roles for $\mathscr{P}$-coalgebras

Like any monad, we can associate to $\mathscr{P}$ the *Kleisli category* $Set_{\mathscr{P}}$, with objects sets and morphisms $Set_{\mathscr{P}}(X, Y) = Set\big(X, \mathscr{P}(Y)\big)$.

Identity morphisms are defined through η and composition through μ.

Definition. Given a $\mathscr{P}$-coalgebra (V, α), we define the **semigroup of roles** $Role_{\mathscr{P}}(\alpha)$ as the subsemigroup of $Set_{\mathscr{P}}(V, V)$ generated by

$\alpha : V \to \mathscr{P}(V)$.

Uh-oh!

Iterated Covariant Powerset is not a Monad [1]

Bartek Klin [2]

Faculty of Mathematics, Informatics, and Mechanics
University of Warsaw
Warsaw, Poland

Julian Salamanca [3]

Faculty of Mathematics, Informatics, and Mechanics
University of Warsaw
Warsaw, Poland

We don't really need a monad

Let $T\colon Set \to Set$ be a functor and $\mu\colon TT \Longrightarrow T$ a natural transformation such that the following diagram commutes:

$$\begin{tikzcd} TTT \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu_T"'] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

which we call an **associative multiplication**.

The **Kleisli semi-category** Set_T has objects sets and morphisms $Set_T(X,Y) = Set(X,T(Y))$. Composition is defined through μ.

Tying the two analyses together

Theorem [Motamed, O, Roff 2025]

Let $T\colon \mathit{Set} \to \mathit{Set}$ be a functor equipped with an associative multiplication. The assignment of semigroup of roles extends to a functor

$$\mathrm{Role}\colon \mathrm{Coalg}_{\mathrm{surj}}(T) \to \mathrm{SemiGroup}_{\mathrm{surj}}\,.$$

In particular, any blockmodel of a social system modelled by a T-algebra induces a quotient of the associated semigroup of roles.

Coalgebraic analysis of social systems, N. Motamed, O., E. Roff

Multiplications on $\mathscr{P}\mathscr{P}$

In 2018 John Baez asked on the n-Category Cafe. . .

Question. Does there exist an associative multiplication $m\colon P^2P^2 \Rightarrow P^2$? In other words, is there a natural transformation $m\colon P^2P^2 \Rightarrow P^2$ such that

$$P^2P^2P^2 \xRightarrow{mP^2} P^2P^2 \overset{m}{\Rightarrow} P^2$$

equals

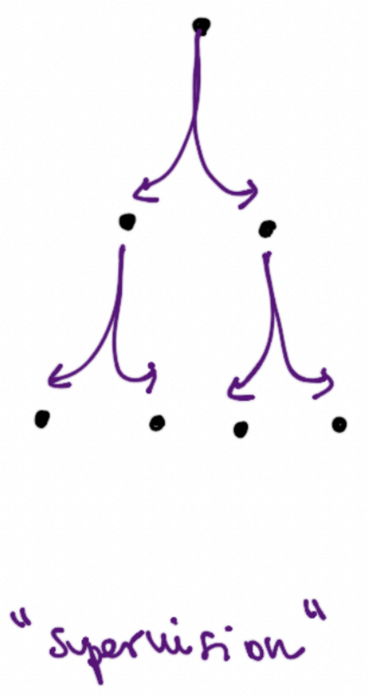
$$P^2P^2P^2 \xRightarrow{P^2m} P^2P^2 \overset{m}{\Rightarrow} P^2.$$

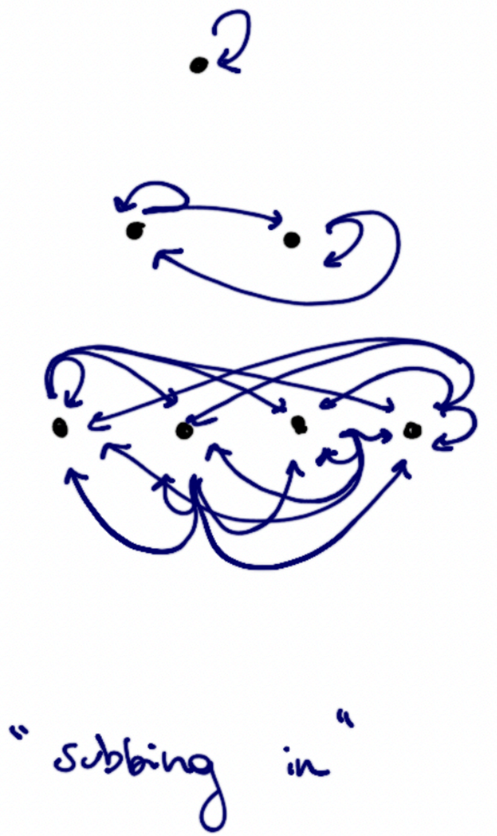
Greg Egan answered: Yes! There are at least two:

$$\mu_1 : \mathcal{PPPP} \xRightarrow{\mu\mathcal{PP}} \mathcal{PPP} \xRightarrow{\mu\mathcal{P}} \mathcal{PP} \quad \text{and} \quad \mu_2 : \mathcal{PPPP} \xRightarrow{\mathcal{PP}\mu} \mathcal{PPP} \xRightarrow{\mathcal{P}\mu} \mathcal{PP}$$

where μ is the multiplication of the monad $\mathcal{P}$.

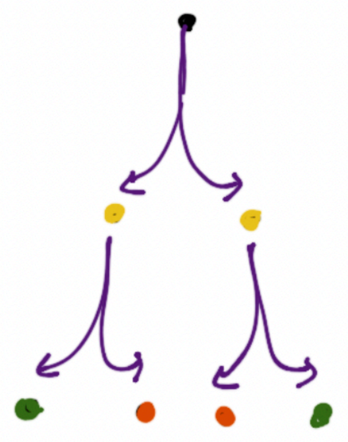
Back to our running example: relationships between employees in a firm





Just “supervision” (S)

Example of partition from a bisimulation equivalence:



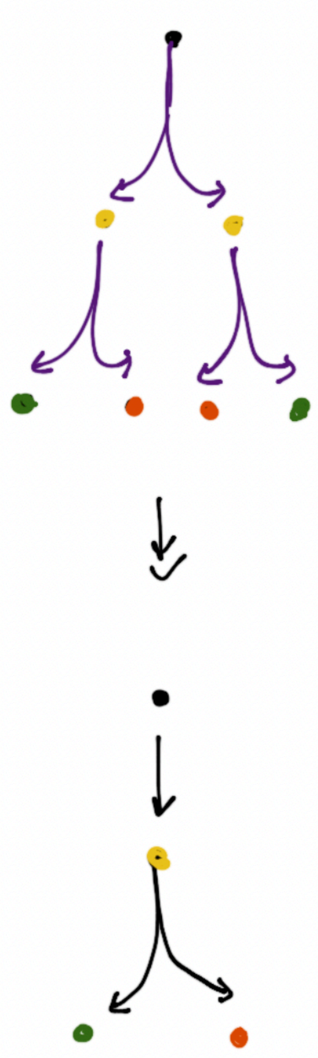
Multiplication 1:

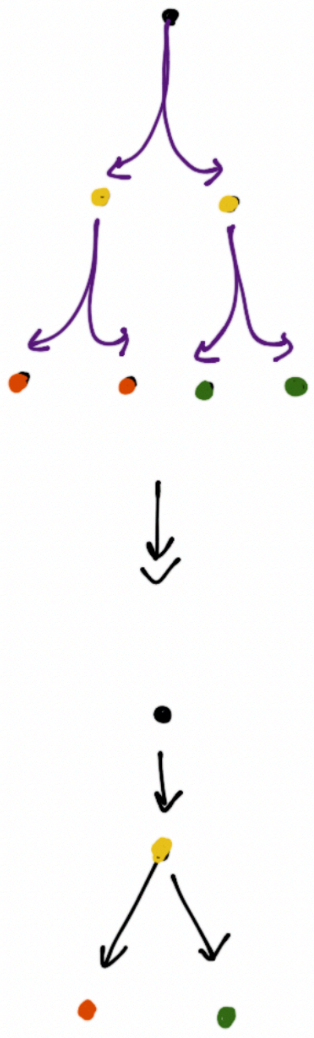
	S	S;S	S;S;S
S	S;S	S;S;S	S;S;S
S;S	S;S;S	S;S;S	S;S;S
S;S;S	S;S;S	S;S;S	S;S;S

Multiplication 2:

	S	S;S	S;S;S
S	S;S	S;S;S	S;S;S
S;S	S;S;S	S;S;S	S;S;S
S;S;S	S;S;S	S;S;S	S;S;S

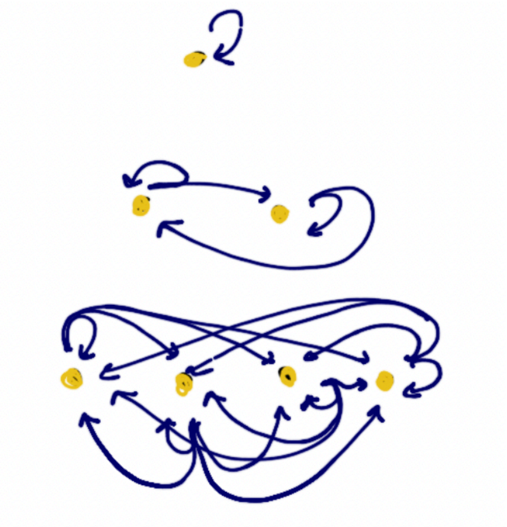
Note: every blockmodel of a hypergraph induces a blockmodel of underlying graph, however, it contains more refined information.





Just “subbing in” (B)

Example of bisimulation equivalence (maximal one):



Multiplication 1:

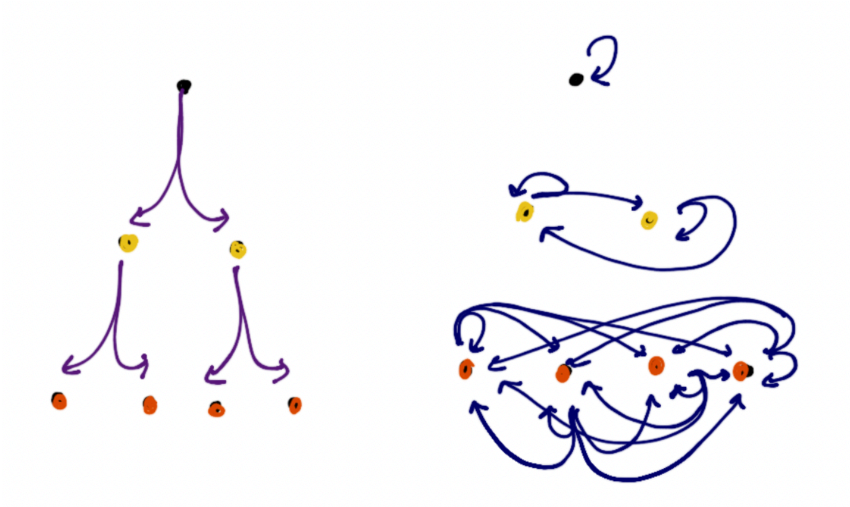
	B
B	B

Multiplication 2:

	B
B	B

The two combined

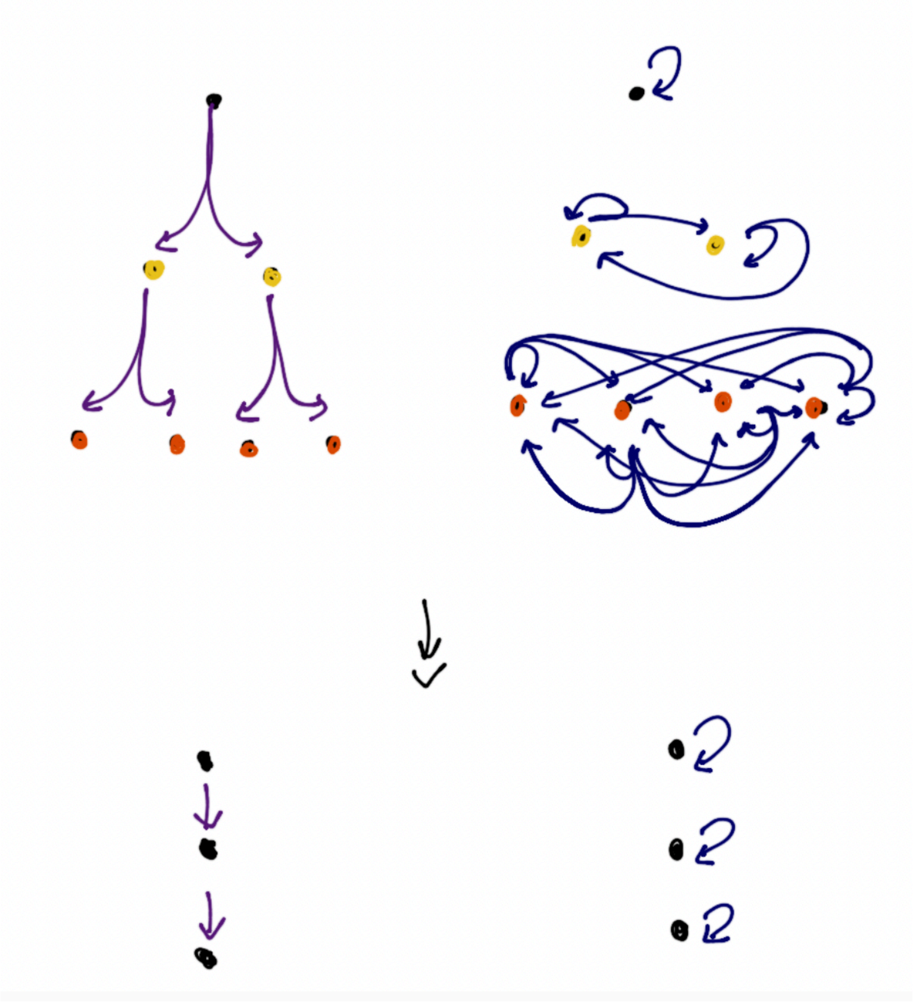
Maximal partition from bisimulation equivalence:



Multiplication 1: 7 elements

Multiplication 2: 8 elements

Functoriality theorem in action (for mult. 1)



$\mapsto$

	B	S	S;B	S;S	B;S	S;S;S	S;S;B
B	B	B;S	S;B	S;S	B;S	S;S;S	S;S;B
S	S;B	S;S	S;S;B	S;S;S	S;S	S;S;S	S;S;S
S;B	S;B	S;S	S;S;B	S;S;S	S;S	S;S;S	S;S;S
S;S	S;S;B	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S
B;S	S;B	S;S	S;S;B	S;S;S	S;S	S;S;S	S;S;S
S;S;S	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S
S;S;B	S;S;B	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S	S;S;S

$\mapsto$

	S	B	S;S	S;S;S
S	S;S	S	S;S;S	S;S;S
B	S	B	S;S	S;S;S
S;S	S;S;S	S;S	S;S;S	S;S;S
S;S;S	S;S;S	S;S;S	S;S;S	S;S;S

The objective

Generalise, to hypergraphs~~/simplicial complexes~~:

- Positional analysis ✔
- Role analysis ✔
- Tie the analyses together through a functoriality result ✔

Future work and Open problems

- Easy-to-use software for practitioners
- Suite of methods to analyse lattices of bisimulations and congruences in semigroups of relations
- General hypergraphs
- Approximate equivalences

Some references

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- A unified framework for equivalences in social networks, N.O., M.A. Porter, 2020
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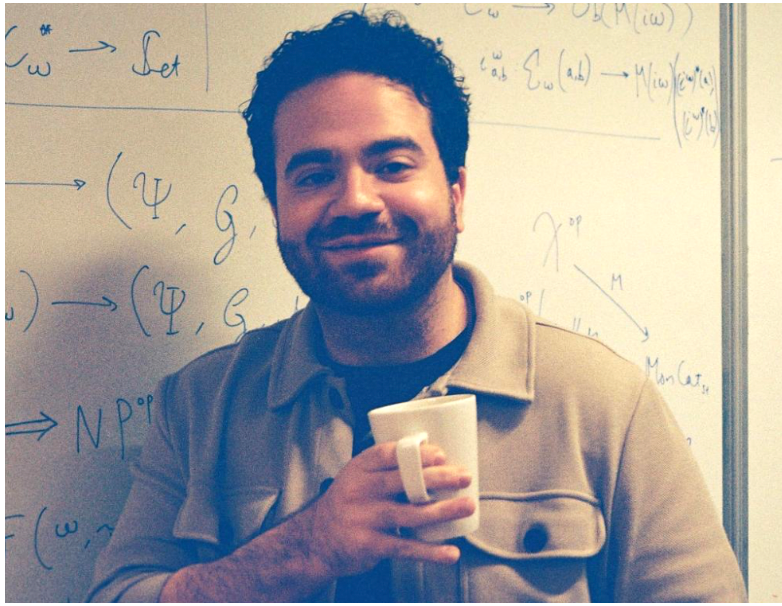
- Universal coalgebra: a theory of systems, Rutten, *Theoretical Computer Science*, 2000

..joint work with



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Thank you!