$\mathscr{V}$-graded categories as a setting for enrichment and actions of monoidal categories $\mathscr{V}$

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- $\mathscr{V}$-graded categories subsume both $\mathscr{V}$-enriched categories and $\mathscr{V}$-actegories [Wood].

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R. B. B. Lucyshyn-Wright, *$\mathscr{V}$-graded categories and $\mathscr{V}$-$\mathscr{W}$-bigraded categories: Functor categories and bifunctors over non-symmetric bases*, Preprint (2025). arXiv:2502.18557

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A **right $\mathscr{V}$-actegory** is a left $\mathscr{V}^{\mathsf{rev}}$-actegory, whose associated functor $\mathscr{C} \times \mathscr{V} \to \mathscr{C}$ we write as a right action $(A, X) \mapsto A.X$.

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Equivalently, a (left) $\mathscr{V}$-graded category $\mathscr{C}$ consists of

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The 2-category of (left) $\mathscr{V}$-graded categories:

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Every $\mathscr{V}$-graded category $\mathscr{C}$ has an *underlying ordinary category* $\mathscr{C}_0$ with the same objects, Every $\mathscr{V}$-graded category $\mathscr{C}$ has an *underlying ordinary category* $\mathscr{C}_0$ with the same objects, in which a morphism $f : A \to B$ is a graded morphism $f : I, A \to B$ whose grade is I.

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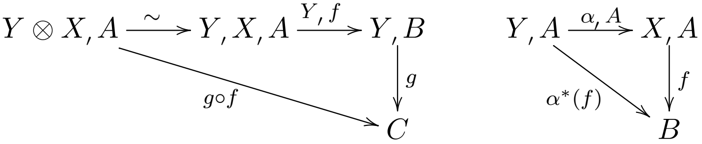

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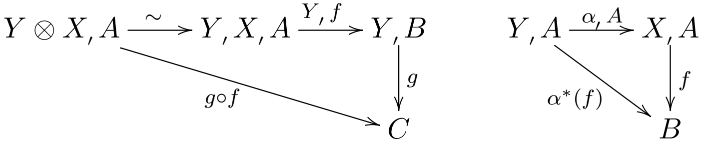

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Every $\mathscr{V}$-$\mathscr{W}$-biactegory may be regarded as a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

Example. A $\mathscr{V}$-$\mathscr{W}$-**biactegory** is a left $(\mathscr{V} \times \mathscr{W}^{\mathsf{rev}})$-actegory, whose associated functor $\mathscr{V} \times \mathscr{C} \times \mathscr{W} \to \mathscr{C}$ is written as a two-sided action $(X, A, Y) \mapsto X.A.Y$.

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Example. $\mathscr{V}$ is a $\mathscr{V}$-$\mathscr{V}$-biactegory

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Example. $\mathscr{V}$ is a $\mathscr{V}$-$\mathscr{V}$-biactegory and so may be regarded as a $\mathscr{V}$-$\mathscr{V}$-bigraded category. Similarly, $\hat{\mathscr{V}}$ is a $\mathscr{V}$-$\mathscr{V}$-biactegory.

Graded transformations

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category,

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$$\begin{array}{ccc} X, FA, Y & \xrightarrow{X, \phi_A} & X, GA \\ {\scriptstyle Ff, Y}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB, Y & \xrightarrow{\phi_B} & GB \end{array}$$

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Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category. Given left $\mathscr{V}$-graded functors $F, G : \mathscr{A} \rightrightarrows \mathscr{C}$ and an object $Y \in \mathrm{ob}\,\mathscr{W}$, a **graded transformation** $\phi : F, Y \Rightarrow G$ is a family of graded morphisms $\phi_A : FA, Y \to GA$ $(A \in \mathrm{ob}\,\mathscr{A})$ in $\mathscr{C}$ that are **left $\mathscr{V}$-graded natural in** $A \in \mathscr{A}$ in the sense that the following envelope diagram commutes for every graded morphism $f : X, A \to B$ in $\mathscr{A}$:

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Given instead a right $\mathscr{W}$-graded category $\mathscr{B}$

Graded transformations

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category. Given left $\mathscr{V}$-graded functors $F, G : \mathscr{A} \rightrightarrows \mathscr{C}$ and an object $Y \in \mathrm{ob}\,\mathscr{W}$, a **graded transformation** $\phi : F, Y \Rightarrow G$ is a family of graded morphisms $\phi_A : FA, Y \to GA$ $(A \in \mathrm{ob}\,\mathscr{A})$ in $\mathscr{C}$ that are **left $\mathscr{V}$-graded natural in** $A \in \mathscr{A}$ in the sense that the following envelope diagram commutes for every graded morphism $f : X, A \to B$ in $\mathscr{A}$:

$$\begin{array}{ccc} X, FA, Y & \xrightarrow{X,\,\phi_A} & X, GA \\ {\scriptstyle Ff, Y}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB, Y & \xrightarrow{\phi_B} & GB \end{array}$$

Given instead a right $\mathscr{W}$-graded category $\mathscr{B}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$,

Graded transformations

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category. Given left $\mathscr{V}$-graded functors $F, G : \mathscr{A} \rightrightarrows \mathscr{C}$ and an object $Y \in \mathrm{ob}\,\mathscr{W}$, a **graded transformation** $\phi : F, Y \Rightarrow G$ is a family of graded morphisms $\phi_A : FA, Y \to GA$ $(A \in \mathrm{ob}\,\mathscr{A})$ in $\mathscr{C}$ that are **left $\mathscr{V}$-graded natural in** $A \in \mathscr{A}$ in the sense that the following envelope diagram commutes for every graded morphism $f : X, A \to B$ in $\mathscr{A}$:

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Given instead a right $\mathscr{W}$-graded category $\mathscr{B}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we can similarly define **graded transformations** $\phi : X, F \Rightarrow G$

Graded transformations

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category. Given left $\mathscr{V}$-graded functors $F, G : \mathscr{A} \rightrightarrows \mathscr{C}$ and an object $Y \in \mathrm{ob}\,\mathscr{W}$, a **graded transformation** $\phi : F, Y \Rightarrow G$ is a family of graded morphisms $\phi_A : FA, Y \to GA$ $(A \in \mathrm{ob}\,\mathscr{A})$ in $\mathscr{C}$ that are **left** $\mathscr{V}$**-graded natural in** $A \in \mathscr{A}$ in the sense that the following envelope diagram commutes for every graded morphism $f : X, A \to B$ in $\mathscr{A}$:

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Given instead a right $\mathscr{W}$-graded category $\mathscr{B}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we can similarly define **graded transformations** $\phi : X, F \Rightarrow G$ between right $\mathscr{W}$-graded functors $F, G : \mathscr{B} \rightrightarrows \mathscr{C}$.

Graded functor categories

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

Graded functor categories

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Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category,*

Graded functor categories

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$*

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$ are the objects of a right $\mathscr{W}$-graded category ${}^{\mathscr{V}}[\mathscr{A}, \mathscr{C}]_{\mathscr{W}}$*

Graded functor categories

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$ are the objects of a right $\mathscr{W}$-graded category ${}^{\mathscr{V}}[\mathscr{A},\mathscr{C}]_{\mathscr{W}}$ that we denote also by $[\mathscr{A},\mathscr{C}]$,*

Graded functor categories

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

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Graded functor categories

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Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$ are the objects of a right $\mathscr{W}$-graded category ${}^{\mathscr{V}}[\mathscr{A},\mathscr{C}]_{\mathscr{W}}$ that we denote also by $[\mathscr{A},\mathscr{C}]$, in which a graded morphism is a graded transformation.*
2. *If $\mathscr{B}$ is a right $\mathscr{W}$-graded category,*

Theorem

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1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$ are the objects of a right $\mathscr{W}$-graded category ${}^{\mathscr{V}}[\mathscr{A},\mathscr{C}]_{\mathscr{W}}$ that we denote also by $[\mathscr{A},\mathscr{C}]$, in which a graded morphism is a graded transformation.*
2. *If $\mathscr{B}$ is a right $\mathscr{W}$-graded category, then right $\mathscr{W}$-graded functors from $\mathscr{B}$ to $\mathscr{C}$*

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$ are the objects of a right $\mathscr{W}$-graded category ${}^{\mathscr{V}}[\mathscr{A},\mathscr{C}]_{\mathscr{W}}$ that we denote also by $[\mathscr{A},\mathscr{C}]$, in which a graded morphism is a graded transformation.*
2. *If $\mathscr{B}$ is a right $\mathscr{W}$-graded category, then right $\mathscr{W}$-graded functors from $\mathscr{B}$ to $\mathscr{C}$ are the objects of a left $\mathscr{V}$-graded category ${}_{\mathscr{V}}[\mathscr{B},\mathscr{C}]^{\mathscr{W}}$*

Graded functor categories

Theorem

Let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

1. *If $\mathscr{A}$ is a left $\mathscr{V}$-graded category, then left $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$ are the objects of a right $\mathscr{W}$-graded category ${}^{\mathscr{V}}[\mathscr{A},\mathscr{C}]_{\mathscr{W}}$ that we denote also by $[\mathscr{A},\mathscr{C}]$, in which a graded morphism is a graded transformation.*
2. *If $\mathscr{B}$ is a right $\mathscr{W}$-graded category, then right $\mathscr{W}$-graded functors from $\mathscr{B}$ to $\mathscr{C}$ are the objects of a left $\mathscr{V}$-graded category ${}_{\mathscr{V}}[\mathscr{B},\mathscr{C}]^{\mathscr{W}}$ that we denote also by $[\mathscr{B},\mathscr{C}]$.*

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Example. Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category. Then $\mathscr{V}$ and $\hat{\mathscr{V}}$ are $\mathscr{V}$-$\mathscr{V}$-bigraded categories, so we obtain right $\mathscr{V}$-graded categories

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The latter is isomorphic to Street's $\hat{\mathscr{V}}$-enriched presheaf category $\mathcal{P}(\mathscr{A})$

Graded functor categories

Example. Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category. Then $\mathscr{V}$ and $\hat{\mathscr{V}}$ are $\mathscr{V}$-$\mathscr{V}$-bigraded categories, so we obtain right $\mathscr{V}$-graded categories

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Example. Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category. Then the formal opposite $\mathscr{A}^\circ$ is a *right* $\mathscr{V}$-graded category, so we obtain left $\mathscr{V}$-graded categories

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Graded bifunctors

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Graded bifunctors

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, let $\mathscr{B}$ be a right $\mathscr{W}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category.

Graded bifunctors

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, let $\mathscr{B}$ be a right $\mathscr{W}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category. A **($\mathscr{V}$-$\mathscr{W}$-)graded bifunctor** $F : \mathscr{A}, \mathscr{B} \to \mathscr{C}$ consists of

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that agree on objects, such that for all $f : X, A \to A'$ in $\mathscr{A}$ and $g : B, Y \to B'$ in $\mathscr{B}$

Graded bifunctors

Let $\mathscr{A}$ be a left $\mathscr{V}$-graded category, let $\mathscr{B}$ be a right $\mathscr{W}$-graded category, and let $\mathscr{C}$ be a $\mathscr{V}$-$\mathscr{W}$-bigraded category. A **($\mathscr{V}$-$\mathscr{W}$-)graded bifunctor** $F : \mathscr{A}, \mathscr{B} \to \mathscr{C}$ consists of

1. left $\mathscr{V}$-graded functors $F(-, B) : \mathscr{A} \to \mathscr{C}$ $(B \in \mathrm{ob}\, \mathscr{B})$ and
2. right $\mathscr{W}$-graded functors $F(A, -) : \mathscr{B} \to \mathscr{C}$ $(A \in \mathrm{ob}\, \mathscr{A})$

that agree on objects, such that for all $f : X, A \to A'$ in $\mathscr{A}$ and $g : B, Y \to B'$ in $\mathscr{B}$ the following envelope diagram commutes:

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Graded bifunctors $F : \mathscr{A}, \mathscr{B} \to \mathscr{C}$ are the objects of a category ${}_{\mathscr{V}}\mathsf{GBif}_{\mathscr{W}}(\mathscr{A}, \mathscr{B}; \mathscr{C})$.

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Graded bifunctors and bigraded products

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$$ {}_{\mathscr{V}}\mathsf{GCAT}_{\mathscr{W}}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C}) \;\cong\; {}_{\mathscr{V}}\mathsf{GBif}_{\mathscr{W}}(\mathscr{A}, \mathscr{B}; \mathscr{C}) $$

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2-natural in $\mathscr{A} \in {}_{\mathscr{V}}\mathsf{GCAT}$, $\mathscr{B} \in \mathsf{GCAT}_{\mathscr{W}}$, $\mathscr{C} \in {}_{\mathscr{V}}\mathsf{GCAT}_{\mathscr{W}}$.

Graded functor categories and bifunctors

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