Combinatorial Foundation for Planar String Diagrams

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PART I

Molecules: The Shapes of Diagrams

cfr: my book

arχiv: 2404.07273

Def In a poset, $\Delta x := \{ y \mid x \text{ covers } y \}$

FACES of x

In a graded poset,

- $\dim x = 0$ iff x is minimal,
- $\dim x = k > 0$ iff

  1. x has a face,
  2. $\forall y \in \Delta x, \ \dim y = k-1$

Def An oriented graded poset is

- a graded poset P,
- for each $x \in P$, a bipartition

$$\Delta x = \underset{\substack{\uparrow \\ \text{OUTPUT}}}{\Delta^{+}} x + \underset{\substack{\uparrow \\ \text{INPUT}}}{\Delta^{-}} x$$

Def A morphism of ogps is a function $f : P \longrightarrow Q$ inducing $\forall x \in P$, $\forall \alpha \in \{+,-\}$, a bijection $\Delta^{\alpha} x \xrightarrow[f_*]{\sim} \Delta^{\alpha} f(x)$

Fact There are operators

$$\partial_n^+, \quad \partial_n^- \quad \text{on ogps}, \quad \forall n \in \mathbb{N}$$

OUTPUT n-BOUNDARY INPUT n-BOUNDARY

Def An ogp U is

- *globular* if $k < n \Rightarrow \partial_k^\alpha \partial_n^\beta U = \partial_k^\alpha U$
- *round* if, furthermore, $\forall k < \dim U$,

$$\partial_k^+ U \cap \partial_k^- U = \partial_{k-1}^+ U \cup \partial_{k-1}^- U =: \partial_{k-1} U$$

Def The class of molecules is inductively generated by:

1. 1, the point, is a molecule
2. if U, V are molecules, $\partial_k^+ U \simeq \partial_k^- V$, the pasting $U \#_k V$ is a molecule
3. if U, V are round molecules, $\dim U = \dim V$, $\partial^\alpha U \simeq \partial^\alpha V$, the rewrite $U \Rightarrow V$ is a molecule

Def An **atom** is a molecule with a greatest element.

Fact Molecules are **rigid**: they have no non-trivial automorphisms.

We can treat isomorphic molecules as equal (cfr H.-Kessler 2022)

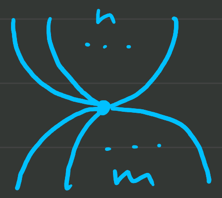
DIM 0: only one atom $\bullet$

DIM 1: one molecule $\forall n > 0$

$$n\vec{I} := \bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \; \cdots \; \bullet \xrightarrow{n} \bullet$$

ALWAYS ROUND

DIM 2: one atom $\forall$ pairs $n, m > 0$



one molecule $\forall$ planar, regular string diagram

ROUND IFF CONNECTED

Fact (Iso-classes of) molecules form a strict ω-category $\mathcal{M}ol$ with

1. ∂_k^{α} as k-boundary operators,
2. $\#_k$ as k-composition operator

Fact So do iso-classes in the slice $\mathcal{M}ol/_P$ for all ogp P

Def An ogp P is a **regular directed complex** if $\forall x \in P$ the lower set $\mathrm{cl}\{x\}$ is an atom.

Fact Let $f : P \longrightarrow Q$ be an order-preserving map of rdcpx s. TFAE:

a) f is a morphism of ogps;

b) f is a local embedding of ogps

$$\left(f|_{\mathrm{cl}\{x\}} : \mathrm{cl}\{x\} \xrightarrow[\text{iso of ogps}]{\sim} \mathrm{cl}\{f(x)\} \quad \forall x \in P \right)$$

Fact There is an OFS on **Pos**

$$\begin{tikzcd} & f_* P \arrow[dr, "f_{\mathcal{L}}"] & \\ P \arrow[ur, "f_{\mathcal{F}}"] \arrow[rr, "f"] & & Q \end{tikzcd}$$

$\mathcal{F}$ final map ← FINAL FUNCTOR

$\mathcal{L}$ local embedding ← DISCRETE GROTH. FIBRATION

"Def" $p : P \longrightarrow Q$ order-preserving map of rdcpx s

is a **map** iff

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d, "f"'] & p_* U \arrow[d, dashed, "p_* f"] \\ P \arrow[r, "p"'] & Q \end{tikzcd}$$

determines a functor

$$\mathrm{Mod}/P \xrightarrow{p_*} \mathrm{Mod}/Q \; .$$

Def A map $p : P \longrightarrow Q$ of rdcpxs is cartesian if it is a Grothendieck fibration of the underlying posets.

$\underline{\mathrm{RdCpx}}$:= regular directed complexes & cartesian maps

$\cup$

$\odot$:= full subcat on atoms

ATOM

RdCpx has a ternary factorisation system:

1. cartesian final map,
2. surjective local embedding,
3. inclusion (i.e. embedding)

On ⊙ 2 is trivial, so we only get an OFS

COLLAPSE — INCLUSION

(epi) (mono)

PART II

Diagrams in Diagrammatic Sets

cfr: jww C. Chanavat

arχiv: 2407.06285, 2410.00123

Def A diagrammatic set is a presheaf on $\odot$.

$\odot\underline{\mathrm{Set}}$:= diagrammatic sets & morphisms of presheaves

Fact The yoneda embedding factors as

$$\begin{tikzcd} \odot \arrow[rr, hook] \arrow[dr, hook] & & \odot\underline{\mathrm{Set}} \\ & \underline{\mathrm{RdCpx}} \arrow[ur, hook, "\text{FULL \& FAITHFUL}"'] & \end{tikzcd}$$

Def Let U be a rdcpx, X a dgm set.

A diagram of shape U in X is a morphism $u : U \longrightarrow X$.

A diagram is a

- pasting diagram if U is a molecule,
- round diagram if U is a round molecule,
- cell if U is an atom.

Def $u : U \longrightarrow X$ cell is nondegenerate if

$$\begin{tikzcd} U \arrow[rr, "u"] \arrow[dr, "p"'] & {} \arrow[d, phantom, "\parallel"] & X \\ & V \arrow[ur, "v"'] & \end{tikzcd} \quad \Longleftrightarrow \quad \begin{array}{l} p = \mathrm{id}, \\ u = v \end{array}$$

$\mathrm{nd}\, X$:= nondegenerate cells in X

$\mathrm{dgn}\, X := \mathrm{cell}\, X \setminus \mathrm{nd}\, X$

Proposition (Eilenberg-Zilber property)

Let $u : U \longrightarrow X$ be a cell.

Then $\exists!$ $p_u : U \twoheadrightarrow V$ collapse,

$\mathrm{supp}(u) : V \longrightarrow X$ nondegenerate

s.t. $u = \mathrm{supp}(u) \circ p_u$.

Consequence Every dgmset is "dimensionwise-free" on its nondegenerate cells

Def Let $u : U \longrightarrow X$ be a diagram of shape U.

The combinatorial diagram associated with u

is the pair of

① the rdcpx U,

② the function $\ell(u) : U \longrightarrow \mathrm{nd}\, X$,

$$x \longmapsto \mathrm{supp}(u|_{\mathrm{cl}\{x\}}).$$

Proposition

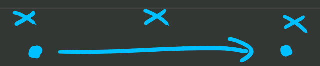
Let $u, v : U \longrightarrow X$ be diagrams.

Then $u = v \iff \ell(u) = \ell(v)$.

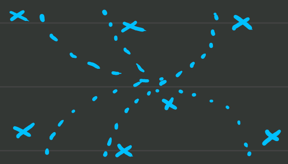
COMBINATORIAL DIAGRAMS ARE A FAITHFUL, FINITE, COMPUTATIONAL ENCODING OF DIAGRAMS

DIM 0 : NO DEGENERATE CELLS

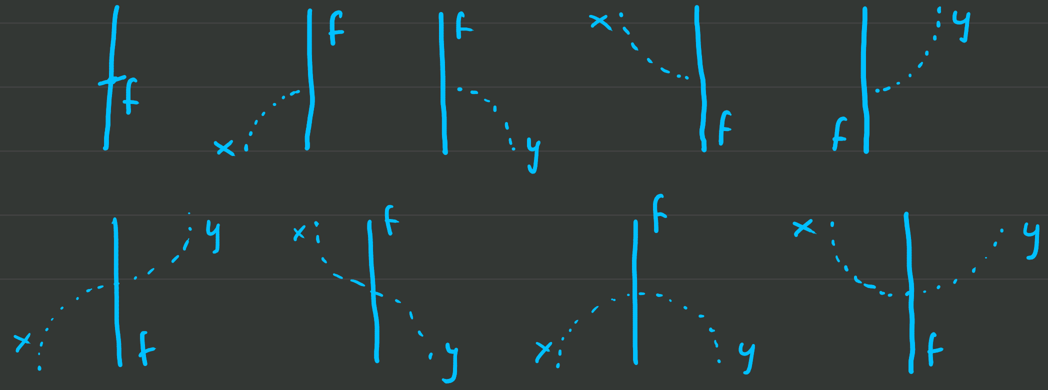
DIM 1 : The only degenerate cells are units



DIM 2 : Degenerate cells are of the types



DOUBLY DEGENERATE



Fact Pasting diagrams in a dgmset X also form a strict ω-category with

① $\partial^\alpha_k (u : U \longrightarrow X) := u|_{\partial^\alpha_k U} : \partial^\alpha_k U \longrightarrow X$

as k-boundary operators,

②
$$\begin{tikzcd}
& U \arrow[dr] \arrow[drr, bend left, "u"] & & \\
\partial^+_k U = \partial^-_k V \arrow[ur] \arrow[dr] & \arrow[r, phantom, "<"] & U \#_k V \arrow[r, dashed, "u \#_k v"] & X \\
& V \arrow[ur] \arrow[urr, bend right, "v"'] & &
\end{tikzcd}$$

as k-composition

"**Def**" A round context (ON (v, w), PARALLEL ROUND DIAGRAMS) in a dgmset X is a "round diagram with a round hole" (OF TYPE (v, w))

Fact Every round context can be put in the form

$$\ell_k \#_{k-1} \left(\dots \ell_1 \#_0 \boxed{} \#_0 r_1 \dots \right) \#_{k-1} r_k$$

with $\dim \ell_i$, $\dim r_i \le i$.

PART III

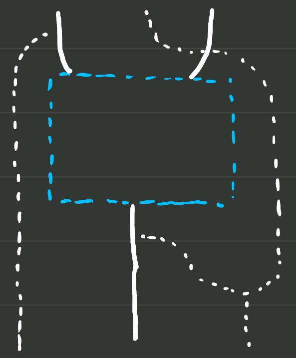
Combinatorial String Diagrams

work in progress

Def A padding is a round context s.t. there is a decomposition $(\ell_i, r_i)_{i=1}^{k}$ where

$\forall i$ $\forall$ cell x of dimension i in ℓ_i, r_i

x is degenerate.

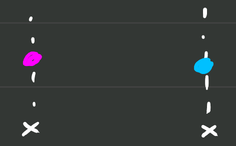


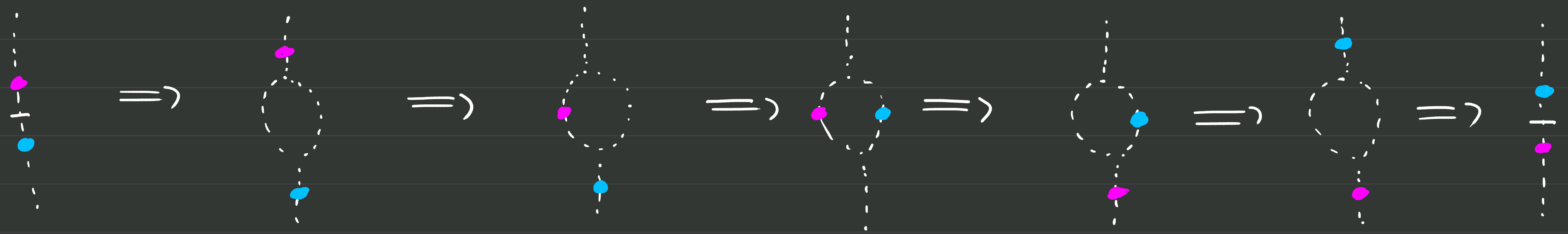
Def Let u, v be round n-dimensional diagrams in a dgmset X.

A **combinatorial isotopy** from u to v is a pair of

1. a padding E on $(\partial^- u, \partial^+ u)$,
2. an $(n+1)$-dimensional pasting diagram $h : E[u] \Rightarrow v$ of degenerate $(n+1)$-cells.

Example (Eckmann-Hilton)

Let  be 2-cells with boundaries degenerate on the same 0-cell x.



Fact $u \sim v$ iff

$\exists$ a combinatorial isotopy from

u to v is an equivalence relation

between round diagrams.

Moreover, $u \sim v$ implies $\partial_k^\alpha u \sim \partial_k^\alpha v$.

Let X be a dgm set; we can construct the free 2-category $\mathbb{F}_2(X)$ on the nondegenerate 0, 1, 2-cells of X.

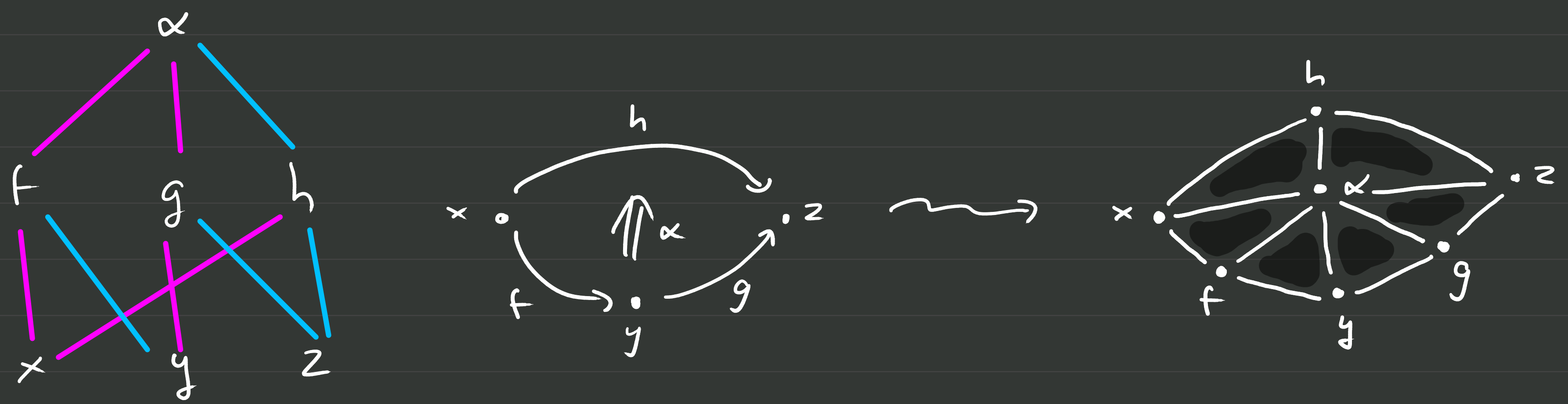
Proposition $\forall k \in \{0, 1, 2\}$,

$$k\text{-cells in } \mathbb{F}_2(X) \overset{\sim}{\longleftrightarrow} \frac{\text{round } k\text{-diagrams in } X}{\text{combinatorial isotopy}}$$

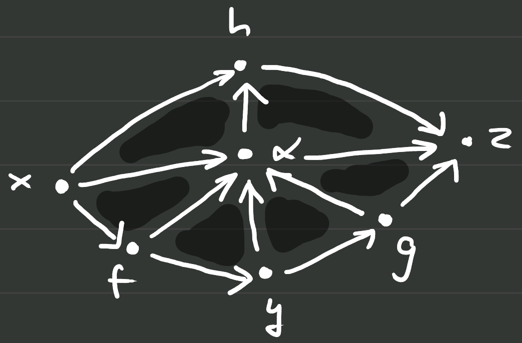
a combinatorial Joyal-Street theorem

We can <u>realise</u> every round 2-diagram as a "Joyal-Street" string diagram, in such a way that the presentation of a free monoidal category by round 2-diagrams / combinatorial isotopy factors through the presentation by JS string diagrams / isotopy

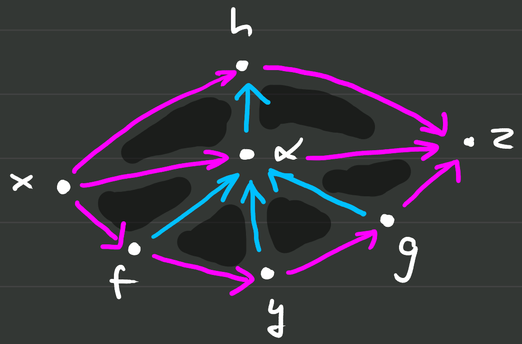
1. Pass from a round molecule to its order complex, a 2d simplicial complex



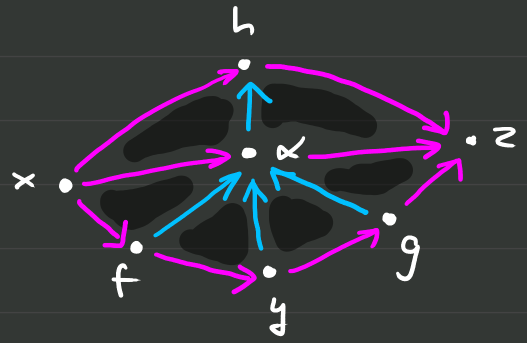
② There is a canonical direction on edges ...



... as well as a bipartition in horizontal vs vertical edges.



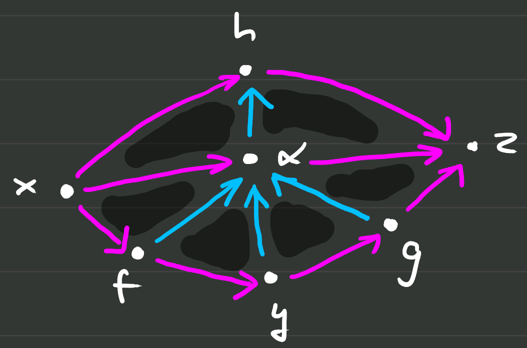
"Horizontal" vs "vertical" paths determine "disjoint" partial orders on the vertices, whose union is a total order.

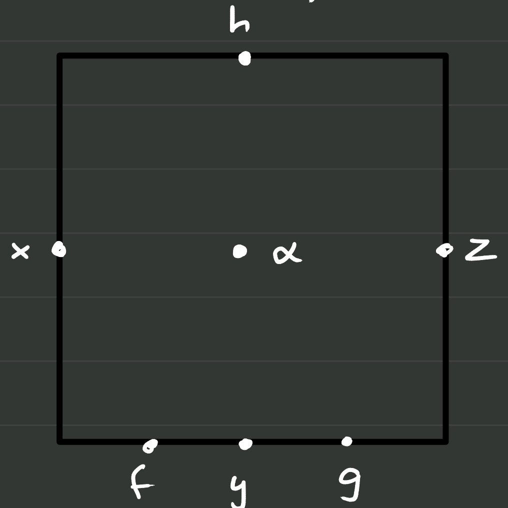


③ Give coordinates to each vertex v by

$$\frac{\text{length of longest } \substack{\text{horizontal} \\ \text{vertical}} \text{ path ending in } v}{\text{length of longest } \substack{\text{horizontal} \\ \text{vertical}} \text{ path passing through } v}$$

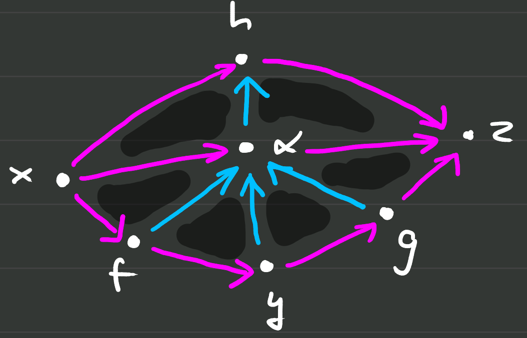
(or $\frac{1}{2}$ if the denominator is 0)

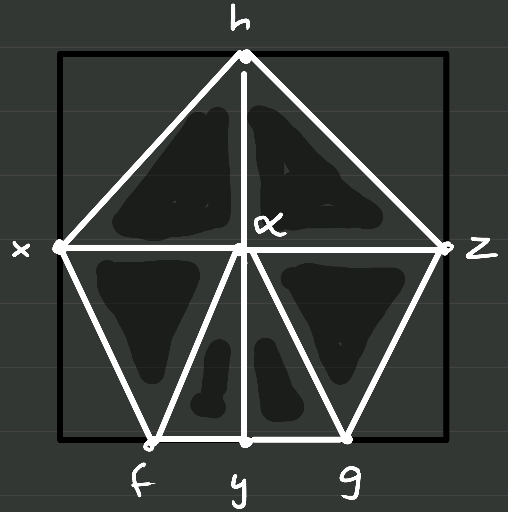




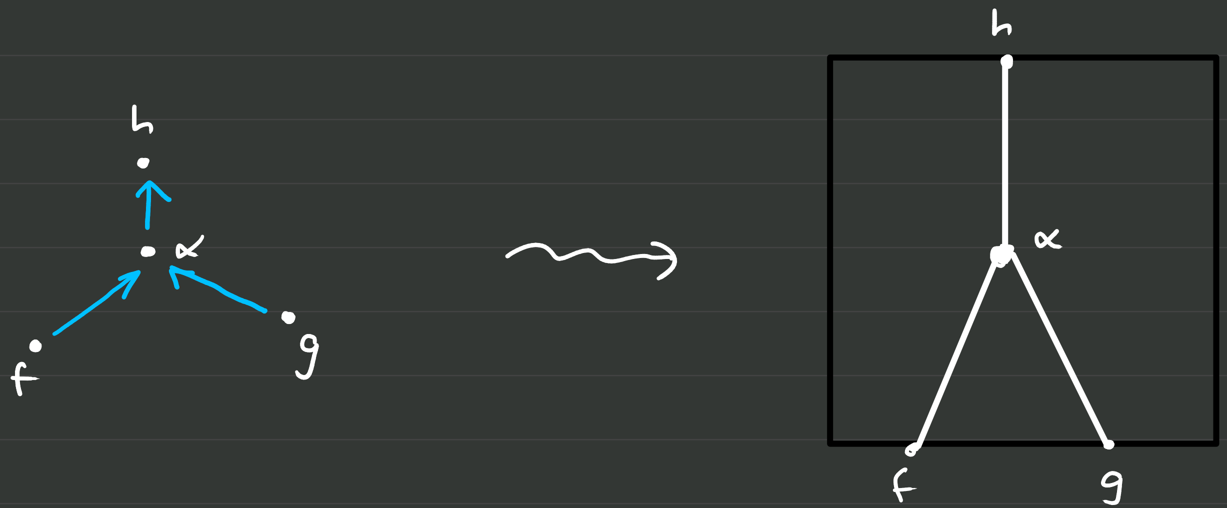
(4) Extend linearly along convex combinations to higher simplices.

This embeds the complex into $[0,1]^2$.



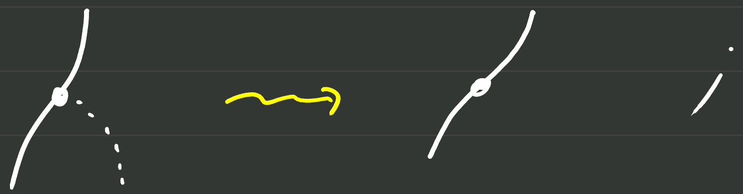


5 Restrict to vertices of dimension 1, 2.

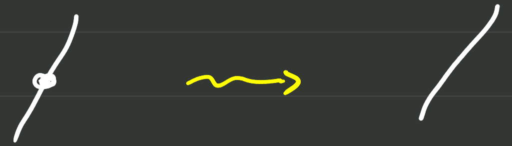


⑥ Finally,

- if a 1d vertex maps to a degenerate 1-cell, omit it & all edges passing through it



- if a 2d vertex maps to a degenerate 2-cell, omit it & merge its incoming & outgoing edges, if present



Consequence

Every isotopy between JS string diagrams in the image of this realisation <u>can</u> be lifted to a combinatorial isotopy.

EPILOGUE

(∞, n)-Categorical String Diagrams

Fact There is a full and faithful nerve functor

$$\underline{\mathrm{Bicat}} \xrightarrow{N} \odot\underline{\mathrm{Set}}$$

BICATEGORIES & STRONG FUNCTORS

This justifies the use of diagrammatic reasoning for (non-strict!) bicategories...

The image of this functor lands in diagrammatic $(\infty, 2)$-categories, the fibrant objects of a model structure for $(\infty, 2)$-categories.

More in general, $\odot\underline{Set}$ supports models for (∞, n)-categories up to $n = \infty$, satisfying the homotopy hypothesis.

Thus:

- string-diagrammatic proofs are sound for (∞, n)-categories up to some combinatorial isotopy;
- if the proof is formalised explicitly in terms of combinatorial isotopies of round 2-diagrams, it produces explicit 3d homotopies.

Thanks for listening!

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