Random Variables, Independence Structures and Dagger Categories of Relations

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Outline

Section 1 – Model

Random Variables and Probability Sheaves

Section 2 – Generalize

Markov categories $\mathbb{C}$, probability spaces $\mathbb{P}(\mathbb{C})$, sample spaces $\mathbb{S}(\mathbb{C})$
Examples: isometries, nondeterminism, fresh name generation

Section 3 – Understand

What's the relationship between the categories $\mathbb{P}$ and $\mathbb{S}$?
Generalizing the regular category $\leftrightarrow$ tabular allegory equivalence

Section 1

Random Variables

Random variables

Let's desugar the following statement

"Let X, Y be independent standard normal variables, then $P(X \geq Y) = \frac{1}{2}$"

1. There exists a sample space $(\mathbf{\Omega}, \mathbf{\Sigma}, \mathbf{P})$ and two measurable functions

$$X, Y : (\mathbf{\Omega}, \mathbf{\Sigma}) \to (\mathbb{R}, \mathcal{B})$$

2. The laws $P_X(A) = P(X^{-1}(A)), P_Y(A) = P(Y^{-1}(A))$ satisfy

$$P_X(A) = P_Y(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{1}{2}x^2} \mathrm{d}x$$

3. Independence

$$P(\{\omega : X(\omega) \in A, Y(\omega) \in B\}) = P_X(A) \cdot P_Y(B)$$

4. Then conclude that $P(\{\omega : X(\omega) \geq Y(\omega)\}) = \frac{1}{2}$

Random variables

What's nice about random variables?

1. Can be manipulated like values
2. measures are constructed implicitly by pushforward
3. close connection with functional analysis $[L^p(\Omega, \Sigma, P)]$
4. $X = Y$ equality almost surely
5. conditional expectation
6. **dependence on Ω implicit**

Random variables

What's awkward about random variables?

1. **dependence on Ω implicit**
2. type-safety: $\mathbb{E}[(X - \mathbb{E}[X])^2]$, $\mathbb{E}[X|Y = y]$, $\mathbb{E}[X|Y]$, ...
3. constructing explicit distributions
4. $X \overset{d}{=} Y$ equality in distribution
5. conditional distributions

Questions [Tao]

1. What is the formal status of the sample space (Ω, Σ, P)?
2. How can we silently enlarge it (allocate new random variables)?
3. How to make sure everything stays consistent?

A Convenient Setting for Random Variables

Convenient Setting

Can we find a typed setting which

1. needs no explicit tracking of measurability or sample spaces
2. faithfully includes standard Borel spaces X
3. has an object $\mathrm{RV}(X)$ of random variables valued in X
4. allows reasoning by higher-order logic?

A Convenient Setting for Random Variables

Some desiderata

1. $(X \to Y) \to (\mathsf{RV}(X) \to \mathsf{RV}(Y))$
2. $\mathsf{RV}(X \times Y) \cong \mathsf{RV}(X) \times \mathsf{RV}(Y)$
3. $\mathrm{Law} : \mathsf{RV}(X) \to \mathcal{G}(X)$ ← Giry monad
4. $(\perp) \subseteq \mathsf{RV}(X) \times \mathsf{RV}(Y)$
5. $(\sim) \subseteq \mathsf{RV}(X) \times \mathcal{G}(X)$
6. $\mathbb{E} : \mathsf{RV}([0,1]) \to [0,1]$
7. $\mathbb{E}[-|-] : \mathsf{RV}([0,1]) \times \mathsf{RV}(X) \to \mathsf{RV}([0,1])$
8. $\forall F : \mathsf{RV}(X) \forall \mu : \mathcal{G}(Y) \exists G : \mathsf{RV}(Y) : G \sim \mu \wedge G \perp F$

A Convenient Setting for Random Variables

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[Simpson'17] There is a boolean topos satisfying these desiderata, namely **Probability Sheaves**

Probability Sheaves

Probability Sheaves

Definition

Let $\mathbb{SBP}$ be the category of **standard Borel sample spaces**

1. objects are standard Borel probability spaces $\Omega = (X, \Sigma, P)$
2. morphisms $(X, \Sigma_X, P) \to (Y, \Sigma_Y, Q)$ are equivalence classes $[f]_P$ of measure-preserving measurable functions

$$(X, \Sigma_X) \to (Y, \Sigma_Y), f_* P = Q$$

   up to p-almost sure equality.

Interpretation: Morphisms $\pi : \Omega' \to \Omega$ are projections or coarse-grainings, e.g.

1. $(X \times Y, \Sigma_{X \times Y}, P) \to (X, \Sigma_X, P_X)$
2. $(X, \mathcal{F}, P) \to (X, \mathcal{E}, P|_{\mathcal{E}})$ where $\mathcal{E} \subseteq \mathcal{F}$

Probability Sheaves

Definition

A *probability presheaf* is a functor $F : \mathbb{SBP}^{op} \to \mathbf{Set}$.

Idea: Every element $X \in F(\Omega)$ can be **extended** along $\Omega' \xrightarrow{\pi} \Omega$ to $X \cdot \pi \in F(\Omega')$.

Definition

For every standard Borel space V, we have a presheaf $\mathrm{RV}(V) : \mathbb{SBP}^{op} \to \mathbf{Set}$ with

$$\mathrm{RV}(V)(\Omega, p) = \{X : \Omega \to V \text{ measurable } \}/\text{p-a.s.}$$

The extension action is $(\Omega \xrightarrow{X} V) \cdot (\Omega' \xrightarrow{\pi} \Omega) = X \circ \pi$.

Proposition

RV defines a cartesian functor $\mathbf{Sbs} \to [\mathbb{SBP}^{op}, \mathbf{Set}]$.

1. $\mathrm{RV}(V \xrightarrow{f} W)_\Omega : (\Omega \xrightarrow{X} V) \mapsto f \circ X$
2. the desired operations are equivariant (natural transformations), e.g.

$$\mathrm{Law} : \mathrm{RV}(V) \to \Delta\mathcal{G}(V), (\Omega \xrightarrow{X} V) \mapsto X_* p_\Omega$$

Probability Sheaves

Theorem

Each presheaf $\mathrm{RV}(V)$ is a sheaf for the atomic topology on $\mathbb{SBP}$.

Given $\pi : \Omega' \to \Omega$,

1. $X \cdot \pi = Y \cdot \pi \Rightarrow X = Y$
2. $X' \in \mathrm{RV}(V)(\Omega')$ descends to $X \in \mathrm{RV}(V)(\Omega)$ iff X' is π-invariant, i.e.

$$\forall \rho_1, \rho_2, \pi \circ \rho_1 = \pi \circ \rho_2 \Rightarrow X' \cdot \rho_1 = X' \cdot \rho_2$$

What's the deeper meaning of the atomic sheaf property?

Sheaves and Conditional Independence

Definition (Independent Square)

We call a commutative square in $\mathbb{SBP}$ *independent*

$$\begin{array}{ccc} \Omega' & \xrightarrow{f_1} & \Omega_1 \\ {\scriptstyle f_2}\downarrow & \perp\!\!\!\perp & \downarrow{\scriptstyle g_1} \\ \Omega_2 & \xrightarrow[g_2]{} & \Omega \end{array}$$

if f_1 and f_2 are conditionally independent given $g_1 f_1$ $(= g_2 f_2)$. ← regular conditional probabilities

Theorem (Simpson)

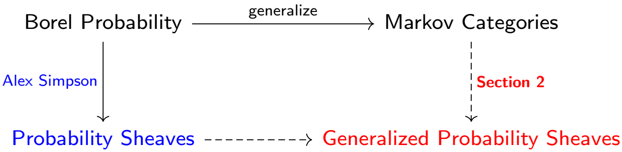
TFAE for a presheaf $F : \mathbb{SBP}^{\mathrm{op}} \to \mathbf{Set}$

1. *F is a sheaf for the* **atomic topology** *on* $\mathbb{SBP}$
2. *F sends independent squares in* $\mathbb{SBP}$ *to pullback squares in* $\mathbf{Set}$

Probability Sheaves

Summary Section I: The random variable formalism lives in **atomic sheaves over sample spaces**.

We want to: Generalize the situation, and understand what makes it work.



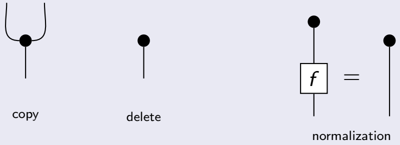
Section 2

Markov Categories and Sample Spaces

Markov categories

Markov categories [Fritz'20]

Markov categories $(\mathbb{C}, \otimes, I, \mathrm{copy}, \mathrm{del})$ are an axiomatization of stochastic maps (Markov kernels)



Markov categories: Examples

FinStoch (discrete probability)

Finite sets X, and stochastic matrices $p(y|x)$ (Kleisli maps $X \to D(Y)$)

BorelStoch (Borel probability)

Standard Borel spaces (X, Σ_X) and Markov kernels (Kleisli maps $(X, \Sigma_X) \to \mathcal{G}(Y, \Sigma_Y)$)

Gauss (Gaussian probability)

Euclidean spaces $\mathbb{R}^n$, and affine-linear maps with Gaussian noise $f(x) = Ax + \mathcal{N}(\mu, \Sigma)$

SetMulti (nondeterminism)

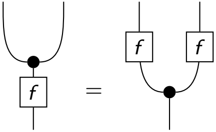
Sets X, and left-total relations $R \subseteq X \times Y$ (Kleisli maps $X \to \mathcal{P}_{\supset\emptyset}(Y)$)

StrongName (fresh name generation)

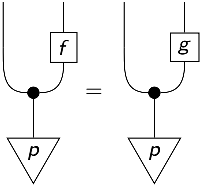
Strong nominal sets X, and name-generating equivariant functions (Kleisli maps $X \to N(Y)$); e.g. $f(a) = \langle\rangle a$, $g(a) = \langle b \rangle b$.

Markov categories

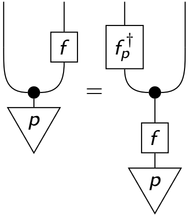
Many probabilistic concepts can be captured precisely in the language of Markov categories:



f deterministic



$f = g$ p-almost surely



Bayesian inversion

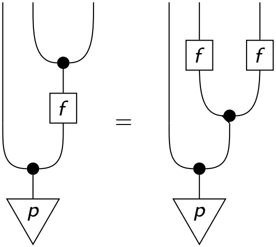
Probability- and Sample Spaces

Let $\mathbb{C}$ be a Markov category with conditionals.

Definition

A probability space is a pair (X, p) with $p : I \to X$. A morphism $(X, p) \to (Y, q)$

- in $\mathbb{P}(\mathbb{C})$ is $[f]_p : X \to Y$ with $fp = q$
- in $\mathbb{S}(\mathbb{C})$ is $[f]_p : X \to Y$ with $fp = q$ and f is p-a.s. deterministic



Classifying Sample Spaces

We wish to understand $\mathbb{S}(\mathbb{C})$ in our examples:

Definition (Simplifying assumption)

Call a sample space (X, p) *faithful* if $f =_p g \Leftrightarrow f = g$. ← forget about a.s. equality

Theorem

The following are equivalent

1. (X, p) *is isomorphic in* $\mathbb{P}(\mathbb{C})$ *to a faithful sample space* (S, σ)
2. (X, p) *is isomorphic in* $\mathbb{S}(\mathbb{C})$ *to a faithful sample space* (S, σ)
3. (S, σ) *is a split support for* p.

In all examples *except* **BorelStoch**, every sample space is isomorphic to a faithful one.

Probability

$\mathbb{P}$(**FinStoch**) is equivalent to the category of couplings

- objects are (X, p) with $p(x) > 0$
- morphisms $(X, p_X) \to (Y, p_Y)$ are joint distribution $p(x, y)$ with $p_1(x) = p_X(x)$, $p_2(y) = p_Y(y)$.

$\mathbb{S}$(**FinStoch**) is equivalent to **FinRV**

- objects are (X, p) with $p(x) > 0$
- morphisms $(X, p) \to (Y, q)$ are surjective functions f with

$$q(y) = \sum_{x \in f^{-1}(y)} p(x)$$

Relations

$\mathbb{P}$(**SetMulti**) is equivalent to

- objects are sets X
- morphisms are bi-total relations $R \subseteq X \times Y$

$\mathbb{S}$(**SetMulti**) is equivalent to **Surj**

- objects are sets X
- morphisms are surjective functions

Gaussian Probability

$\mathbb{P}$(**Gauss**) is equivalent to **Con**

- objects are euclidean spaces $\mathbb{R}^n$
- morphisms are matrices $A \in \mathbb{R}^{n \times m}$ with $||Ax|| \leq ||x||$ (contractions)!

$\mathbb{S}$(**Gauss**) is equivalent to **CoIso**

- objects are euclidean spaces $\mathbb{R}^n$
- morphisms are $A \in \mathbb{R}^{n \times m}$ with $AA^T = I_n$.

Name Generation

$\mathbb{S}(\textbf{StrongName})$ is equivalent to $\textbf{FinInj}^{\mathrm{op}}$

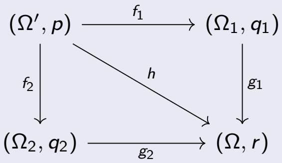
- objects are finite sets n
- morphisms $m \to n$ are injections $n \to m$

$$\textbf{Nom}(\mathbb{A}^{\#m}, \mathbb{A}^{\#n}) \cong \textbf{Inj}(n, m)$$

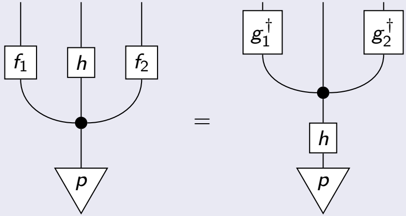
Conditional independence

Definition

We call a commutative square in $\mathbb{S}(\mathbb{C})$



independent if $f_1 \perp f_2 \mid h$, i.e.



Probability- and Sample Spaces

Can we develop a theory of probability sheaves for $\mathbb{S}(\mathbb{C})$?

1. Yes, Simpson postulated axioms (IP1)-(IP5) which can be verified by hand ([Stein, LICS'25])
2. more elegant route: let's understand the relationship between $\mathbb{S}(\mathbb{C})$ and $\mathbb{P}(\mathbb{C})$ using categorical logic

Some Properties [Perrone & al]

1. $\mathbb{P}(\mathbb{C})$ and $\mathbb{S}(\mathbb{C})$ are both semicartesian monoidal (not Markov!)
2. Bayesian inversion is a dagger functor on $\mathbb{P}(\mathbb{C})$

$$(X, p) \underset{f^\dagger}{\overset{f}{\rightleftarrows}} (Y, q)$$

3. f p-a.s. deterministic $\Leftrightarrow f \circ f^\dagger = \mathrm{id}$ ($\dagger$-coisometry!)
4. That is $\mathbb{S}(\mathbb{C}) = \mathrm{CoIsom}(\mathbb{P}(\mathbb{C}))$.

Idea: If $\mathbb{P}$ has a nice enough $\dagger$-structure, then $\mathrm{CoIsom}(\mathbb{P})$ has a nice independence structure.

Section 3

Independence Structures

Quick Recap

Classic story

locally regular categories ↔ **tabular allegories**

New story

epiregular independence categories ↔ **dilar †-categories**

Regular Categories and Allegories

Rel-construction

Every locally regular category $\mathbb{C}$ has a category of relations $\mathrm{Rel}(\mathbb{C})$.

1. same objects as $\mathbb{C}$
2. morphisms are equivalence classes of jointly monic spans $[f \xleftarrow{f} \Omega \xrightarrow{g} Y]$
3. composition by pullback, followed by image factorization

Question: How to characterize the categories $\mathrm{Rel}(\mathbb{C})$, and recover $\mathbb{C}$ from them?

Regular Categories and Allegories

Allegory

An **allegory** is a poset-enriched †-category $\mathbb{R}$ where each hom-set carries a lattice structure $R \cap S$, satisfying the modular law.

1. a **map** $f : X \to Y$ satisfies $1 \subseteq f^\dagger f$ (left-total) and $ff^\dagger \subseteq 1$ (subfunctional).
2. a **tabulation** of R is a span of maps $[f, g]$ with $gf^\dagger = R$
3. a **tabulator** is a tabulation satisfying $f^\dagger f \cap g^\dagger g = 1$ (think joint monicity)

A **tabular allegory** is one where every morphism has a tabulator.

A correspondence

1. For every locally regular category $\mathbb{C}$, $\mathrm{Rel}(\mathbb{C})$ is a tabular allegory
2. For every tabular allegory $\mathbb{R}$, $\mathrm{Map}(\mathbb{R})$ is a locally regular category
3. we have a 2-equivalence

$$\mathbf{LocReg} \underset{\mathrm{Map}}{\overset{\mathrm{Rel}}{\rightleftarrows}} \mathbf{TabAll}$$

Our Solution

For intuitions, keep in mind the following dictionary

Allegories	**Our story**
map	coisometry
tabulation	dilation
tabulator	dilator
pullback	independent pullback

Theorem

1. For every **dilar †-category** $\mathbb{P}$, $\mathrm{CoIsom}(\mathbb{P})$ is an **epiregular independence category**
2. For every **epiregular independence category** $\mathbb{S}$, $\mathrm{Rel}(\mathbb{S})$ is a **dilar †-category**
3. We have a 2-equivalence

$$\mathbf{EpiRegInd} \underset{\mathrm{CoIsom}}{\overset{\mathrm{Rel}}{\rightleftarrows}} \mathbf{DilDag}$$

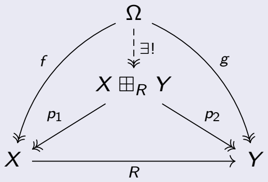
Dilar †-categories

Definition

In a †-category $\mathbb{P}$, we say $f : X \to Y$ is

1. a **coisometry** if $ff^\dagger = 1$ (†-epi)
2. an **isometry** if $f^\dagger f = 1$ (†-mono)

A **dilation** of $R : X \to Y$ is a span of coisometries with $R = gf^\dagger$. A **dilator** is a terminal dilation



We say $\mathbb{P}$ is **dilar** if every morphism has a dilator.

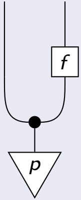
Dilar †-categories

Proposition

If $\mathbb{C}$ is a Markov category with conditionals, then $\mathbb{P}(\mathbb{C})$ is a dilar †-category. The dilator of $f : (X, p) \to (Y, q)$ is given by the span

$$(X, p) \xleftarrow{\pi_1} (X \otimes Y, \rho) \xrightarrow{\pi_2} (Y, q)$$

where

$\rho =$ 

Independence Structures

Theorem

For any †-category $\mathbb{P}$, $\mathrm{CoIsom}(\mathbb{P})$ carries an independence structure

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle g}\downarrow & \perp & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow[v]{} & \bullet \end{array} \quad :\Leftrightarrow \quad fg^\dagger = u^\dagger v$$

Examples:

1. in Hilbert spaces: relative orthogonality
2. in $\mathbb{S}(\mathbb{C})$: conditional independence!

If $\mathbb{P}$ has dilators, this should make the independence structure extra nice ...

Independence Structures

Definition [Alex Simpson'18, ∼]

An *independence category* is equipped with a predicate $\perp$ on commuting squares s.t

(i) $$\begin{array}{ccc} \bullet & \xrightarrow{1} & \bullet \\ f\downarrow & \perp & \downarrow f \\ \bullet & \xrightarrow[1]{} & \bullet \end{array}$$

(ii) $$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ f\downarrow & \perp & \downarrow g \\ \bullet & \xrightarrow[s]{} & \bullet \end{array} \text{ and } \begin{array}{ccc} \bullet & \xrightarrow{b} & \bullet \\ g\downarrow & \perp & \downarrow h \\ \bullet & \xrightarrow[t]{} & \bullet \end{array} \text{ implies } \begin{array}{ccccc} \bullet & \xrightarrow{a} & \bullet & \xrightarrow{b} & \bullet \\ f\downarrow & & \perp & & \downarrow h \\ \bullet & \xrightarrow[s]{} & \bullet & \xrightarrow[t]{} & \bullet \end{array}$$

(iii) $$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ g\downarrow & \perp & \downarrow u \\ \bullet & \xrightarrow[v]{} & \bullet \end{array} \text{ implies } \begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ f\downarrow & \perp & \downarrow v \\ \bullet & \xrightarrow[u]{} & \bullet \end{array}$$

(iv) $$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ f\downarrow & \perp & \downarrow 1 \\ \bullet & \xrightarrow[1]{} & \bullet \end{array}$$

Independence Structures

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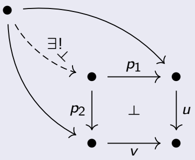
(iv)
$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ f\downarrow & \perp & \downarrow 1 \\ \bullet & \xrightarrow[1]{} & \bullet \end{array}$$

$\leftarrow$ **(iv) not self-dual: independence $\perp$ vs co-independence $\top$**

Independent Pullback

Definition [Simpson]

In an independence category $(\mathbb{S}, \perp)$, an **independent pullback** is terminal among independent squares.



Epiregular independence category

Definition

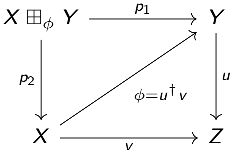
An independence category $(\mathbb{S}, \perp)$ is **epiregular** if

1. every morphism is strong epi (orthogonal to jointly monic spans)
2. every span has a (strong epi, jointly mono)-factorization
3. every span completes to an independent pullback

Theorem

For every dilar †-category $\mathbb{P}$, $\mathrm{CoIsom}(\mathbb{P})$ is an epiregular independence category.

Idea: a span of coisometries $[f, g]$ is jointly monic iff it is a dilator of $gf^\dagger$.



is an independent pullback

Back and Forth

Definition

For an epiregular independence category $(\mathbb{S}, \perp)$, define $\mathrm{Rel}(\mathbb{S}, \perp)$ as usual, but compose by **independent pullback**.

Main Theorem

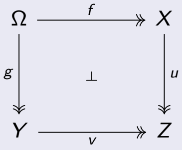
For every epiregular independence category $(\mathbb{S}, \perp)$, $\mathrm{Rel}(\mathbb{S}, \perp)$ is a dilar †-category, and we have a 2-equivalence

$$\mathbf{EpiRegInd} \underset{\mathrm{CoIsom}}{\overset{\mathrm{Rel}}{\rightleftarrows}} \mathbf{DilDag}$$

Examples: Surjections

Conditional variation independence

A commuting square of surjections



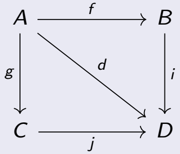
is independent if for all $x \in X$, $y \in Y$ with $u(x) = v(y)$, there exists $\omega \in \Omega$ with $f(\omega) = x$, $g(\omega) = y$.

Here, independence $\Leftrightarrow$ weak pullback in **Set** (but not in Surj).
Independent pullback is the pullback in **Set**.

Examples: Injections

Relative disjointness

A commuting square of injections is co-independent



if $\mathrm{im}(i) \cap \mathrm{im}(j) = \mathrm{im}(d)$.

$\mathrm{Inj}^{\mathsf{op}}$ is an epiregular independence category.

In separation logic, think of Inj as heap layouts. Independence pushouts are **Set**-pushouts, and have been used in the semantic of local state [Kammar&al,LICS'17]

Examples: Isometries of Hilbert spaces

Relative Orthogonality

A bounded linear map f between Hilbert spaces is an isometry if $||fx|| = ||x||$. A commuting square of isometries is co-independent if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & \top & \downarrow{\scriptstyle i} \\ C & \xrightarrow[j]{} & D \end{array} \quad :\Leftrightarrow \quad i^* j = f g^*$$

$\mathrm{Isom}^{\mathsf{op}} \cong \mathrm{CoIsom}$ is an epiregular independence category (essentially by Sz.-Nagy's dilation theorem).

To summarize

We can extended the regular category/allegory 2-equivalence to a 2-equivalence

$$\mathbf{EpiRegInd} \underset{\color{blue}{\mathrm{CoIsom}}}{\overset{\color{red}{\mathrm{Rel}}}{\rightleftarrows}} \mathbf{DilDag}$$

under which $\mathbb{S}(\mathbb{C})$ and $\mathbb{P}(\mathbb{C})$ recover each other. A theory of probability sheaves can be developed for any epiregular independence category $\mathbb{S}$.

$\mathbb{S}(\mathbb{C})$	$\mathbb{P}(\mathbb{C})$	$\mathbb{C}$
FinRV	**Coupl**	**FinStoch**
$\mathbb{SBP}$	**BorelCoupl**	**BorelStoch**
Surj	**TotRel**	**SetMulti**
$\mathbf{Isom}^{op}$	**Con**	**Gauss**
$\mathbf{Inj}^{op}$	**pInj**	**StrongName**

Outlook

Outlook

1. let $\mathcal{E}$ be the topos of atomic sheaves on $\mathbb{S}$
2. the inclusion $J : \mathbb{S} \to \mathbb{P} = \mathrm{Rel}(\mathbb{S})$ induces an adjunction

$$[\mathbb{S}^{\mathsf{op}}, \mathbf{Set}] \underset{\Delta_J}{\overset{\Sigma_J}{\rightleftarrows}} [\mathbb{P}^{\mathsf{op}}, \mathbf{Set}] \qquad (\Sigma_J \dashv \Delta_J)$$

   whose monad restricts to $\mathcal{M} : \mathcal{E} \to \mathcal{E}$ and is commutative and affine
3. the Kleisli category $\mathcal{K}l(\mathcal{M})$ is a Markov category
4. if $\mathbb{S} = \mathbb{S}(\mathbb{C})$, then $\mathbb{C} \to \mathcal{K}l(\mathcal{M})$ a Markov embedding.

Application in Computer Science: "A Nominal Approach to Probabilistic Separation Logic" [Li&al, LICS'24] (Lilac), "A Monad for Full Ground Reference Cells" [Kammar&al, LLICS'17]

Appendix

The 2-category **DilDag** consists of

0. dilar †-categories
1. functors preserving daggers and dilators
2. natural coisometries

The 2-category **EpiRegInd** consists of

0. epiregular independence categories
1. functors preserving independent squares
2. natural transformations with independent naturality squares