Articulating the Structure of Reasons

Logical Expressivism

and

Implication-Space Semantics

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Full Disclosure

This lecture shamelessly promotes some of the ideas developed in more detail in our book

Reasons for Logic, Logic for Reasons:

Pragmatics, Semantics, and Conceptual Roles

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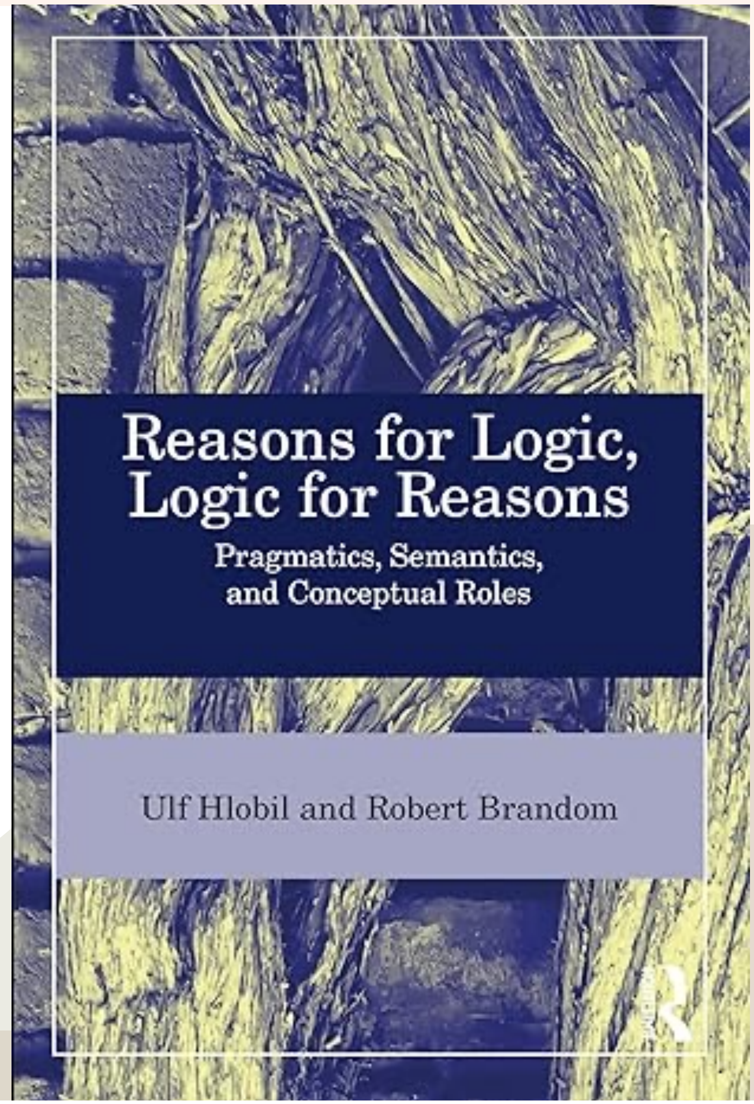

Semantic Inferentialism

- **Project**: Understand the sort of **propositional conceptual content** that consists in the **inferential roles** declarative sentences play as the premises and conclusions of **implication relations**.

- **Question**: What sort of **logic** and **semantics** can we do with propositional contents understood ***relationally***, that is, as conferred on sentences by the consequence relations in virtue of which some express **reasons** for others?

Vocabularies

- A ***lexicon*** L is a set of sentences.

- From the lexicon, we construct the set of all pairs of sets of sentences of L, thought of as the set of ***candidate implications*** of L:

$$\mathrm{CandImp}(L) =_{\mathrm{df.}} \mathcal{P}(L) \times \mathcal{P}(L)$$

If $\langle A,B\rangle \in \mathrm{CandImp}(L)$, the first element $A \subseteq L$ is the ***premise set*** of the candidate implication $\langle A,B\rangle$ and the second element $B \subseteq L$ is the ***conclusion set*** of the candidate implication $\langle A,B\rangle$.

A consequence relation is just a binary relation on the powerset of a lexicon of sentences.

Each ***vocabulary*** on a lexicon is the pair of the lexicon L and some distinguished proper subset **I** of CandImp(L) thought of as the ***good*** implications:

$$\langle L,\mathbf{I}\rangle = \langle L, \mathbf{I} \subseteq \mathcal{P}(L) \times \mathcal{P}(L) \rangle$$

Orienting Question:

What can logic tell us about implication relations?

There is a consensus about the structure of specifically logical consequence relations.

Tarski on Consequence as a Closure Operator

Containment (CO):	$X \subseteq Con(X)$.
Monotonicity (MO):	$X \subseteq Y \Rightarrow Con(X) \subseteq Con(Y)$.
Transitivity-as-Idempotence (CT):	$Con(Con(X)) = Con(X)$.

A Structural Mismatch

Many nonlogical senses of ‘reason for’ (in law, in medicine) are probative, and defeasible, not dispositive.

Materially Good Implications can be Substructural

<u>Nonmonotonic</u>:

Tweety is a bird |~ Tweety flies

but

Tweety is a bird, Tweety is a penguin # Tweety flies.

<u>Nontransitive</u>:

Tweety is a penguin |~ Tweety is a bird and **Tweety is a bird |~ Tweety flies**

but

Tweety is a penguin # Tweety flies

First Idea:

Logic is best understood in terms of its **applications**,

not as **pure** logic.

1. What logic is applied ***to*** is inferential 'theories':

the ***material*** consequence relations of ***non***logical vocabularies.

2. **Logical** rules compute the consequence relation of the vocabulary that results from logically **extending** some **pre**logical **base vocabulary**.

Logics as Functions from Consequence Relations to Consequence Relations

A prelogical base vocabulary $<L_0,\mathbf{I}_0>$ consists of a lexicon L_0 of atomic sentences and a consequence relation $\mathbf{I}_0$, which is a set of pairs of sets of sentences of L, thought of as premises and conclusions of good implications.

The **syntactic** rules of a logic determine a function
from the base lexicon L_0 **to** its logical extension L:
$L_0 \subseteq L$, and if $A,B \in L$, then $A \rightarrow B \in L$, $\neg A \in L$, $A \& B \in L$, and $A \vee B \in L$.

The **semantic** rules determine a function
from the consequence relation $\mathbf{I}_0 \subseteq \mathcal{P}(L_0) \times \mathcal{P}(L_0)$ on the base lexicon L_0
to a consequence relation $\mathbf{I} \subseteq \mathcal{P}(L) \times \mathcal{P}(L)$ on the lexicon L that results from the addition of logically complex sentences.

- **Tarski** made a conceptual advance by thinking topologically about the structure of logical consequence relations.

- **Gentzen** took a further step by treating particular instances of consequence relations as mathematical *objects* he called 'sequents,' whose metainferential relations we can study.

The sequent '$\Gamma \mid\sim \Delta$'

says that

the premise-set Γ ***implies*** the conclusion-set Δ.

Sequent rules are *meta-inferential*.

They say that *if* one set of implications is good,
then so is another.

Tarski's Monotonicity (MO):

$$X \subseteq Y \;\Rightarrow\; Con(X) \subseteq Con(Y)$$

becomes Gentzen's:

$$\frac{\Gamma \mid\!\sim \Delta}{\Gamma, \Theta \mid\!\sim \Delta} \qquad \text{MO}$$

Sequent rules can be applied to two kinds of sequents relating atomic sentences:

1. Those that hold in virtue of their ***structure*** alone, such as

A |~ A

2. Those that are ***materially*** good, in virtue of the contents of the *non*logical concepts they involve, such as

'Berkeley is to the West of Pittsburgh' |~ 'Pittsburgh is to the East of Berkeley'

and

'It is raining' |~ 'The streets will be wet'

Why Do Logic?

What is the *point* of elaborating a material consequence relation governing a prelogical vocabulary into a consequence relation governing a vocabulary consisting of logical compounds of those atomic sentences?

Second Idea:

The *point* of introducing logical vocabulary is not what it lets us ***prove***, but what it lets us ***say***.

- Pure logic can ***prove*** that some implication holds in virtue of logic alone.

  But it is more important that

- **Conditionals *say that* an implication is good**—never mind whether it is *logically* good or *materially* good (good in virtue of the base consequence relation).

Logical Expressivism:

The expressive task distinctive of **logical** vocabulary as such is to **express** reason relations of implication and incompatibility explicitly in the form of the claimable propositional contents of logically complex sentences.

Making Implications Explicit

Deduction-Detachment (DD):

$$\Gamma, A \mid\!\sim B, \Delta$$

$$=========$$

Bidirectional Meta-Inference Line

$$\Gamma \mid\!\sim A \rightarrow B, \Delta.$$

This sequent rule shows how the conditional can be introduced to codify implications.

The conditional A→B is implied by a premise-set Γ just in case if A ***were*** added to Γ, the resulting premise-set ***would*** imply B.

So Γ having A→B as a consequence ***says that*** B follows from A, in the context Γ.

The Expressive Ideal for Logic

1. The ideal logic would conservatively **elaborate** the reason relations of a logically extended vocabulary from those of its nonlogical base vocabulary.

2. The ideal logic would have the expressive power to **make explicit** the reason relations of any base vocabulary, as well as its logical extension.

3. The ideal logic would be ***structurally universal***.

The reason relations codifiable by expressively ideal logical vocabulary include topologically open (nonmonotonic and nontransitive) ones.

In sum, **the expressively ideal logic is *Universally LX***:

- It can be conservatively **Elaborated *from*** (L)
- and is ***Explicative of*** (X)
- **any** constellation of reason relations whatsoever, regardless of its structure: **universally**.

Third Idea:

Compare sets of sequent-calculus connective definitions along **two dimensions**:

1. The **relative expressive power** of the logical connectives they introduce to *say that* various constellations of consequence relations hold.

2. **Substructurally**: The extent to which that expressive power persists (or degrades gracefully) when structural rules such as monotonicity (MO) and transitivity (CT) are *not* imposed.

There are Two Kinds of Sequent Rules:

1. Structural Rules, which do *not* depend on logical connectives occurring in the premises or conclusions of sequents:

$$\frac{\Gamma \mid\!\sim A \qquad \Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta} \qquad \text{(CT)}$$

2. Connective Rules, which *do* depend on the occurrence of logical connectives in the premises or conclusions of sequents:

$$\frac{\Gamma, A \mid\!\sim \Delta}{\Gamma, A\&B \mid\!\sim \Delta}$$

Strategy:

Compare the results of combining different sets of sequent-calculus **connective definitions** with different proper subsets of the **structural rules**.

Look for the **most expressively powerful logical** connective definitions across the **widest variety of substructural** prelogical base vocabularies.

We Have a Winner!

Daniel Scott **Kaplan** realized that tweaking the reversible connective definitions discovered by Gentzen's student Ketonen can yield a remarkably expressively powerful logic, **NMMS**.

It requires only the minimal structure of Containment (CO):

Premises imply themselves in all contexts.

*N*on*M*onotonic, *M*ulti-*S*uccedent Logic (NMMS)

Connective Rules of NMMS:

L¬: $$\frac{\Gamma \mid\!\sim \Delta, A}{\Gamma, \neg A \mid\!\sim \Delta}$$

R¬: $$\frac{\Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta, \neg A}$$

L→: $$\frac{\Gamma \mid\!\sim \Delta, A \quad B, \Gamma \mid \Delta \quad B, \Gamma \mid\!\sim \Delta, A}{\Gamma, A \rightarrow B \mid\!\sim \Delta}$$

R→: $$\frac{\Gamma, A \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A \rightarrow B, \Delta}$$

L&: $$\frac{\Gamma, A, B \mid\!\sim \Delta}{\Gamma, A \& B \mid\!\sim \Delta}$$

R&: $$\frac{\Gamma \mid\!\sim \Delta, A \quad \Gamma \mid\!\sim \Delta, B \quad \Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \& B}$$

L∨: $$\frac{\Gamma, A \mid\!\sim \Delta \quad \Gamma, B \mid\!\sim \Delta \quad \Gamma, A, B \mid\!\sim \Delta}{\Gamma, A \vee B \mid\!\sim \Delta}$$

R∨: $$\frac{\Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \vee B}$$

*N*on*M*onotonic, *M*ulti-*S*ucceedent Logic (NMMS)

NMMS is ***expressively complete*** in a strong sense.

For each **set** of sequents that holds in a base vocabulary, we can compute a **single** sequent relating logically complex sentences that holds in all and only NMMS logical elaborations of base vocabularies in which those atomic sequents hold—and *vice versa.*

In this clear sense, the logically ***complex*** **NMMS** sequent ***says that*** its corresponding logically ***atomic*** sequents hold.

*N*on*M*onotonic, *M*ulti-*S*uccedent Logic (NMMS)

Compare:

In a classical 2-valued truth-functional semantic setting, for any given logically **complex** sentence, it is settled just what combinations of logically **atomic** sentences have to be true for it to be true.

In our setting, for any given **implication** relating logically complex sentences, it is settled just which **implications** relating ***atomic*** sentences must be good for it to be good.

*N*on*M*onotonic, *M*ulti-*S*uccedent Logic (NMMS)

NMMS is ***universally*** LX.

NMMS **conservatively elaborates** even logically atomic base vocabularies that are **nonmonotonic** and **nontransitive** (and more!).

NMMS is **expressively complete** for all base vocabularies satisfying Containment (CO).

*N*on*M*onotonic, *M*ulti-*S*uccedent Logic (NMMS)

NMMS is essentially just **classical logic**.

In the fully structural, topologically closed setting defined by Gentzen's (and Tarski's) full set of structural rules, **NMMS** yields **exactly the same logical consequence relation** as Gentzen's sequent-calculus version of classical logic, **LK**.

NMMS is **supraclassical** when applied to any base vocabularies that include all instances of CO, and

NMMS yields exactly the *classically valid* implications and incompatibilities if it is applied to base vocabularies *all* of whose implications are instances of CO

(a "flat prior").

*N*on*M*onotonic, *M*ulti-*S*uccedent Logic (NMMS)

NMMS is ***not*** a **nonmonotonic logic**.

It is a **logic for codifying nonmonotonic** (and nontransitive) **consequence relations**.

Its own purely *logical* consequence relation is structurally closed and classical: reflexive, monotonic, and transitive.

The consequence relation on the full, logically extended vocabulary is ***not*** closed, since NMMS ***conservatively*** extends ***sub***structural material base vocabularies to which it is applied.

PART 2:
ROLES AND REASONS

Implication-space Semantics
and Propositional Content

Implication Spaces

An *implication space* is the set of all pairs of sets of sentences of a *lexicon* L, thought of as the *candidate implications* of L: $\mathcal{P}(L) \times \mathcal{P}(L)$.

An implication-space *frame* (vocabulary) is the pair of an implication space defined over a lexicon, together with a distinguished subset **I** of that space, thought of as the *good* implications:

$$\langle \mathcal{P}(L) \times \mathcal{P}(L), \mathbf{I} \rangle.$$

Ranges of Subjunctive Robustness

The *range of subjunctive robustness* (RSR) of a candidate implication $<\Gamma,\Delta>$ is the set of additions to its premises (and conclusion) that would make it good, if it is *not* good, or keep it good, if it *is* good.

- A commutative monoid on sets of candidate implications combines pairs of them by pointwise union.
- The RSR of a candidate implication $<\Gamma,\Delta>$ is the pre-image of the set **I** of good implications when the monoid is applied to $<\Gamma,\Delta>$:

$$\{<X,Y>: <\Gamma\cup X,\Delta\cup Y>\in \mathbf{I}\}.$$

Ranges of Subjunctive Robustness

The set **I** partitions candidate implications *extensionally* into good/bad.

The RSR of each candidate implication (good or bad) is its *intension*.

Compare: extensional truth *values* and intensional truth *conditions*.

Implicational Roles

The *implicational role* $\mathcal{R}(<\Gamma,\Delta>)$ of a candidate implication $<\Gamma,\Delta>$ is the set of (sets of) candidate implications that have the same range of subjunctive robustness as $<\Gamma,\Delta>$.

The implications in $\mathcal{R}(<\Gamma,\Delta>)$ can be intersubstituted with each other *salva consequentia*:

that is, without turning a *good* implication (in **I**) into a *bad* one (not in **I**).

Conceptual Roles of Sentences

The *conceptual* (propositional) *content* of sentence [A] for some A∈L is the pair of the implicational roles of its *premissory* and *conclusory* seed implications <A,∅> and < ∅,A>.

RSR(<A, ∅>), the range of subjunctive robustness of <A, ∅>, determines all the good implications in which A appears as a premise.

RSR(< ∅,A>), the range of subjunctive robustness of < ∅,A>, determines all the good implications in which A appears as a conclusion.

$$[A] = <a^{+},a^{-}> = < \mathcal{R}(\{<A, \varnothing>\}), \mathcal{R}(\{< \varnothing,A>\}) >$$

These are the inferential *consequences* and *circumstances* of application of the sentence A.

Ranges and Roles again

- The *range of subjunctive robustness* of implication $\Gamma |\!\sim \Delta$ is the set of all candidate implications that, when pointwise (dual) unioned with $\langle \Gamma, \Delta \rangle$ yield a *good* implication (one in **I**).

$$RSR\langle \Gamma, \Delta \rangle =_{df.} \{\langle X, Y \rangle \in \mathcal{P}(L) \times \mathcal{P}(L) : \langle \Gamma \cup X, \Delta \cup Y \rangle \in \mathbf{I}\}.$$

- The *implicational role* of implication $\Gamma |\!\sim \Delta$ is the equivalence class of (sets of) candidate implications that share its range of subjunctive robustness (intension):

$$\mathcal{R}(\{\langle \Gamma, \Delta \rangle\}) =_{df.} \{\langle X, Y \rangle \in \mathcal{P}(L) \times \mathcal{P}(L) : RSR\langle X, Y \rangle = RSR\langle \Gamma, \Delta \rangle\}.$$

Representing Propositions

We symbolize the implicational role of the sentence A by enclosing A in square brackets:

[A]

We can decompose that into the pair of a premissory implicational role and a conclusory implicational role:

$$[A] = \langle a^+, a^- \rangle$$

Each of those elements can be further decomposed:

$$[A] = \langle a^+, a^- \rangle = \langle \mathcal{R}(\{\langle A, \varnothing \rangle\}), \mathcal{R}(\{\langle \varnothing, A \rangle\}) \rangle$$

Example: Negation

The negation rules of NMMS are just those of standard classical logic (Gentzen's LK):

L¬: $$\frac{\Gamma \mid\sim \Delta, A}{\Gamma, \neg A \mid\sim \Delta}$$

R¬ : $$\frac{\Gamma, A \mid\sim \Delta}{\Gamma \mid\sim \Delta, \neg A}$$

The left rule says that the role of ¬A as *premise* is the same as the role of A as *conclusion*, and

The right rule says that the role of ¬A as *conclusion* is the same as the role of A as *premise*.

The idea is that where the propositional role of A, [A] is $\langle a^+, a^- \rangle$,

negation just **exchanges** premissory and conclusory roles:

$$[\neg A] = \langle a^-, a^+ \rangle$$

Operations on Implicational Roles

Symjunction: $\mathcal{R}(X) \sqcap \mathcal{R}(Y) =_{df.} \mathcal{R}(X \cup Y)$.

Adjunction: $\mathcal{R}(X) \sqcup \mathcal{R}(Y) =_{df.} \mathcal{R}(\{\Gamma \cup \Delta : \Gamma \in X, \Delta \in Y\})$.

Semantic Definitions of Connectives of NMMS

$[A] =_{df.} \langle a^{+}, a^{-} \rangle \qquad [B] =_{df.} \langle b^{+}, b^{-} \rangle$

$\sqcup$ is adjunction of implicational roles, $\sqcap$ is symjunction of implicational roles

$$[\neg A] =_{df.} \langle a^{-}, a^{+} \rangle.$$

$$[A \rightarrow B] =_{df.} \langle a^{-} \sqcap b^{+} \sqcap (a^{-} \sqcup b^{+}),\ a^{+} \sqcup b^{-} \rangle.$$

$$[A \& B] =_{df.} \langle a^{+} \sqcup b^{+},\ a^{-} \sqcap b^{-} \sqcap (a^{-} \sqcup b^{-}) \rangle.$$

$$[A \vee B] =_{df.} \langle a^{+} \sqcap b^{+} \sqcap (a^{+} \sqcup b^{+}),\ a^{-} \sqcup b^{-} \rangle.$$

Sequent Definitions of Connectives in NMMS

L¬:
$$\frac{\Gamma \mid\!\sim \Delta, A}{\Gamma, \neg A \mid\!\sim \Delta}$$

R¬:
$$\frac{\Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta, \neg A}$$

L→:
$$\frac{\Gamma \mid\!\sim \Delta, A \qquad B, \Gamma \mid \Delta \qquad B, \Gamma \mid\!\sim \Delta, A}{\Gamma, A \rightarrow B \mid\!\sim \Delta}$$

R→:
$$\frac{\Gamma, A \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A \rightarrow B, \Delta}$$

L&:
$$\frac{\Gamma, A, B \mid\!\sim \Delta}{\Gamma, A \& B \mid\!\sim \Delta}$$

R&:
$$\frac{\Gamma \mid\!\sim \Delta, A \qquad \Gamma \mid\!\sim \Delta, B \qquad \Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \& B}$$

L∨:
$$\frac{\Gamma, A \mid\!\sim \Delta \qquad \Gamma, B \mid\!\sim \Delta \qquad \Gamma, A, B \mid\!\sim \Delta}{\Gamma, A \vee B \mid\!\sim \Delta}$$

R∨:
$$\frac{\Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \vee B}$$

The Metalogical Correspondence between Implication-Space and Sequent-Calculus Metavocabularies:

1. The *first* element in the semantic clauses corresponds to the ***left*** rule in the sequent calculus.

The *second* element corresponds to the ***right*** rule in the sequent calculus.

2. The roles super-scripted with a "**+**" stem from sentences that occur on the ***left*** in a top sequent.

The roles super-scripted with a "**−**" stem from sentences that occur on the ***right*** in a top sequent.

3. An *adjunction* ⊔ indicates that the adjoined roles stem from sentences in a ***single*** top sequent.

A *symjunction* ⊓ indicates that the symjoined roles stem from sentences that occur in ***different*** top sequents.

Example: Comparing Sequent-Calculus and Implication-Space Formulations of Connective Definitions

Sequent Rule

Implication-Space Definition

L→: $\underline{\Gamma|\!\sim\Delta,A \quad B,\Gamma|\Delta \quad B,\Gamma|\!\sim\Delta,A}$

$\Gamma,A\rightarrow B|\!\sim\Delta\ \Gamma$

$[A\rightarrow B] \quad =_{df.} \quad < a^{-}\sqcap b^{+}\sqcap(a^{-}\sqcup b^{+}),$

$a^{+}\sqcup b^{-} >$

R→: $\underline{\Gamma,A|\!\sim B,\Delta}$

$\Gamma|\!\sim A\rightarrow B,\Delta$

Only look at the premises above the sequent line, and only look at the A's and B's.

On the premise side of the sequent turnstile, they get a +, on the conclusion side, they get a -.

The Initial Question

From a *lexicon* of *sentences* L_0, generate the set of ***candidate implications***

$\mathcal{P}(L_0)x\mathcal{P}(L_0)$.

Pick a distinguished subset of these I_0 as the ***good*** implications: the premise-set implies the conclusion-set.

Q: What **structural** restrictions are there on the capacity of **logical** and **model-theoretic** metalanguages to **express** implication relations in this sense and **manipulate propositions** defined by their implicational roles?

Conclusion: Logic

Traditional connective definitions for classical logic build in **Monotonicity** and **Transitivity**.

The logic **NMMS**—a lightly tweaked version of Ketonen's reversible connective definitions:

- has a **classical purely *logical* consequence relation**,
- is ***supraclassical*** when applied to substructural base vocabularies, which it ***elaborates*** conservatively and
- is **strongly expressively complete**: each sequent in the logically extended language expresses a specific set of atomic sequents, and each set of atomic sequents is expressed by a single sequent relating logically complex sentences.

Conclusion: Implication-Space Semantics

To do model theory for **radically substructural** **base** vocabularies $<\mathcal{P}(L_0)x\mathcal{P}(L_0),\mathbf{I}_0>$ and the **logically extended vocabularies** elaborated from them:

- Define a **commutative monoid** on the implication space: pointwise dual union.
- Sort candidate sets of implications into equivalence classes, accordingly as they share pre-images of the good implications under the monoid operation.
- The implicational roles that result can be combined into pairs of **premissory** and **conclusory roles** of **sentences**.

Logical expressivism and **semantic inferentialism** set strenuous requirements for relations to count as ***consequence*** or ***implication*** relations.

Those relations must be suitably **expressible with *logical* vocabulary**, and the roles sentences play in such relations must yield a ***semantically* tractable notion of propositional contents**—in particular a notion of propositional contents that supports combining them in ***conditionals***, conjunctions, negations, and all the other ways sentential logic makes possible.

Semantic inferentialists begin, not with ***sentences***, but with consequence ***relations among*** sentences.

We have seen that logically tractable implication relations

- that capture various senses of **'reason for'** and
- that confer semantically well-behaved **propositional contents**

can be *much* more various structurally than hitherto supposed.

Conclusion

- We have shown how to use variants of traditional ***proof theory*** and ***model theory***—which normally only work in ***topologically closed*** settings—to codify ***radically substructural*** implication relations.
- **Logical expressivism** is vindicated by the expressively powerful logic **NMMS**, which can make explicit arbitrary material base consequence relations.
- **Semantic inferentialism** is advanced by **implication-space semantics**, which is ***sound*** and ***complete*** for sequent-calculus specifications of logics that metainferentially extend *arbitrary* base vocabularies.

Thank You!