Behavioural Metrics via Functor Lifting – A Coalgebraic Approach

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Overview

Behavioural Equivalences

In the analysis of state-based systems, behavioural equivalences (bisimilarity, trace equivalence, ...) relate states with the same behaviour.

Applications

- Comparing a system with its specification
- Minimizing the state space
- Analysis of model transformations
- Verification of cryptographic protocols (are two protocols equivalent from the point of view of an external observer, a.k.a. the attacker?)

Behavioural Equivalences

This talk: bisimilarity (and generalizations thereof ...)

Bisimulation

Let L be a set of labels and let X be a set of states with transition relation $\rightarrow \subseteq X \times L \times X$ (written $x \xrightarrow{a} x'$).

A symmetric relation $R \subseteq X \times X$ is a bisimulation if for all pairs $(x, y) \in R$ and all states x' with $x \xrightarrow{a} x'$, there exists a state y' with $y \xrightarrow{a} y'$ and $(x', y') \in R$.

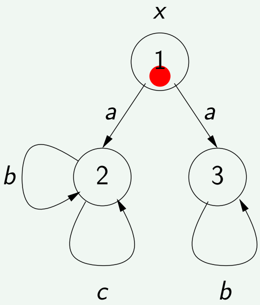
Two states x, y are bisimilar whenever there exists a bisimulation R with $(x, y) \in R$ (written $x \sim y$).

Behavioural Equivalences

Characterization as a two-player game

Two tokens on states x, y – Player I (the attacker) chooses a token and makes a move – Player II (the defender) has to find an answer with the other token – If no answer is possible, Player I wins

Two states are equivalent if and only if there is no winning strategy for Player I.



Behavioural Metrics

Finding a quantitative notion of behavioural equivalence ...

- Do not insist on the exact same behaviour.
- Measure the behavioural distance between two states.
- Make statements such as "the behaviour of two states differs only by ε".

$\rightsquigarrow$ behavioural metrics

Behavioural Metrics

Pseudo-metric space

Let X be a set. A pseudo-metric is a function $d\colon X \times X \to [0,1]$ where for all $x, y, z \in X$:

1. $d(x,x) = 0$ (identity) (metric if $(d(x,y) = 0 \Rightarrow x = y)$)
2. $d(x,y) = d(y,x)$ (symmetry)
3. $d(x,z) \le d(x,y) + d(y,z)$ (triangle inequality)

A (pseudo-)metric space is a pair (X,d) where X is a set and d is a (pseudo-)metric on X.

Behavioural Metrics

Non-expansive function

A non-expansive function $f\colon X \to Y$ between two (pseudo-)metric spaces $(X, d_X), (Y, d_Y)$ satisfies for $x, y \in X$

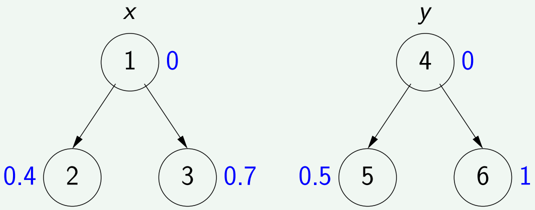
$$d_X(x, y) \geq d_Y(f(x), f(y))$$

Note: the theory can be generalized to quantales (reversing the order).

Metric Transition Systems

Metric transition system [de Alfaro et al., 2009] (slightly simplified)

Let (M, d_r) be a metric space. A metric transition system is a tuple $(X, \tau, [\cdot])$, where X is a set of states, $\tau \subseteq X \times X$ is a transition relation and every state x is assigned an element $[x] \in M$.



Metric space $X = [0, 1]$ with Euclidean metric.

Metric Transition Systems

Hausdorff metric (metric on finite sets)

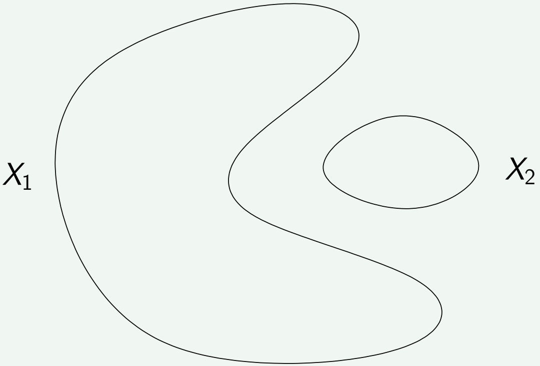
Lifting a metric space (X, d) to $(\mathcal{P}_{fin}X, d')$: for $X_1, X_2 \subseteq X$:

$$d^H(X_1, X_2) = \max\{ \max_{x \in X_1} \min_{y \in X_2} d(x, y),\ \max_{y \in X_2} \min_{x \in X_1} d(x, y) \}$$

- For each element x (in X_1, X_2) take the closest element y in the other set and measure the distance $d(x, y)$
- Take the maximum of all such distances.

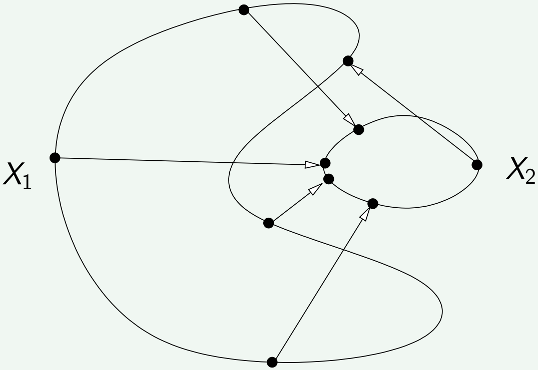
Metric Transition Systems

Example: Hausdorff metric



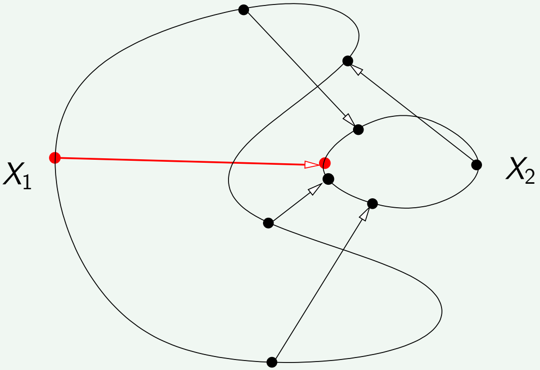
Metric Transition Systems

Example: Hausdorff metric



Metric Transition Systems

Example: Hausdorff metric

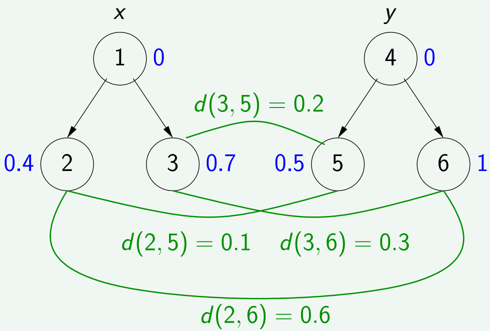


Metric Transition Systems

Distance of states in a metric transition system

Compute the smallest fixed-point of

$$d(x,y) = \max\{\ d_r([x],[y]),\ d^H(\tau(x),\tau(y))\}$$



Metric Transition Systems

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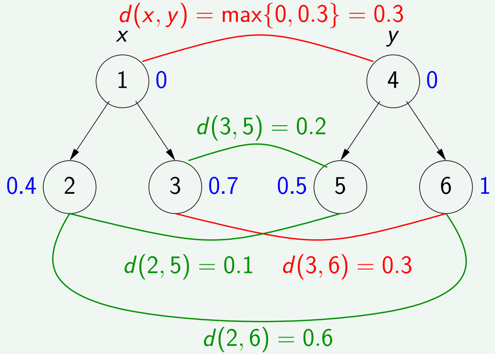


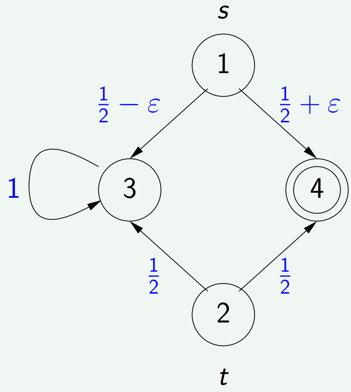
Probabilistic Transition Systems

Probabilistic transition system

A probabilistic transition system is a tuple $(X, T, p_{.})$, where X is a set of states, $T \subseteq X$ is the set of terminal states and every state $x \notin T$ is assigned a probability distribution $p_x \colon X \to [0, 1]$.

Studied by Larsen/Skou [Larsen and Skou, 1989], van Breugel/Worrell [van Breugel and Worrell, 2005] (again simplified)

Probabilistic Transition Systems



Terminal state: 4

What is the distance between states 1 and 2? $\rightsquigarrow$ distance ε

Probabilistic Transition Systems

Distance of states in a probabilistic transition system

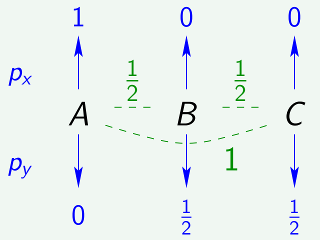
Compute the smallest fixed-point of

$$d(x,y) = \begin{cases} 1 & \text{if } x \in T, y \notin T \text{ or } x \notin T, y \in T \\ 0 & \text{if } x, y \in T \\ d^P(p_x, p_y) & \text{otherwise} \end{cases}$$

What does it mean to compute the distance between two probability distributions p_x, p_y on a metric space?

Transportation Problem & Duality [Villani, 2009]

Lift metric to prob. distr.



distances between states

probabilities of states

Interpret p_x as supply and p_y as demand. Transporting a unit along a distance d costs d.

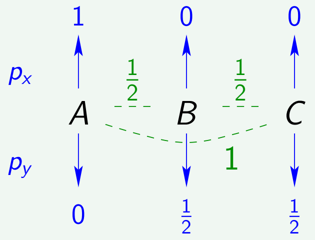
What is the minimal possible cost for transporting supply to demand?

- transport $\frac{1}{2}$ from A to B: cost $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- transport $\frac{1}{2}$ from A to C: cost $1 \cdot \frac{1}{2} = \frac{1}{2}$

Overall cost: $\frac{3}{4}$ ($=$ distance $d^P(p_x, p_y)$)

Transportation Problem & Duality [Villani, 2009]

Alternative: you have a logistics firm and handle transportation. You do this by setting a price (per unit) for locations A, B, C (pr_A, pr_B, pr_C). You buy and sell for this price at every location. Your prices have to satisfy: $pr_B - pr_A \leq d(A, B)$ (otherwise you do not get the contract).



distances between states
probabilities of states

You want to maximize your profit. Which prices do you set?
$\rightsquigarrow$ $pr_A = 0$, $pr_B = \frac{1}{2}$, $pr_C = 1$

- you get: $\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}$
- you pay: $0 \cdot 1 = 0$

Profit: $\frac{3}{4}$

Transportation Problem & Duality [Villani, 2009]

Duality in transportation theory (Kantorovich-Rubinstein duality)

The following values coincide for a metric $d\colon X \times X \to [0,1]$ and two probability distributions $p, q\colon X \to [0,1]$:

The minimum of $\sum_{x,y} P(x,y) \cdot d(x,y)$

for all probability distributions $P\colon X \times X \to [0,1]$ (couplings, indicating transport from x to y), such that $\sum_{y \in X} P(x,y) = p(x)$, $\sum_{x \in X} P(x,y) = q(y)$ (marginal distributions are p, q)

The maximum of $|\sum_{x \in X} f(x) \cdot p(x) - \sum_{x \in X} f(x) \cdot q(x)|$

for all nonexpansive functions $f\colon X \to [0,1]$

Generalization of Metric Transition Systems

Questions:

- How can we model other types of transition systems (with different branching types)?
- How to set up the fixed-point equation in the general case?
- Are there generic (and efficient) methods to compute metrics?

$\rightsquigarrow$ use coalgebra, a general categorical theory of behavioural equivalences, to answer these questions.

Coalgebra offers a toolbox from which transition systems with different branching types can be constructed and analyzed.

Functors

Typical examples of Set-endofunctors used for coalgebras

- (finite) powerset functor $\mathcal{P}_{fin}X = \{Y \mid Y \subseteq X, Y \text{ finite}\}$
- probability distribution functor
  $\mathcal{D}X = \{p\colon X \to [0,1] \mid \sum_{x \in X} p(x) = 1\}$
- product functor $FX = A \times X$ (for a fixed set X)
- coproduct functor (disjoint union) $FX = X + B$ (for a fixed set B)
- combinations of these functors

The functor defines the branching type of the transition system:

- powerset functor $\rightsquigarrow$ non-determinism
- probability distribution functor $\rightsquigarrow$ probabilistic branching
- product functor $\rightsquigarrow$ labelling
- coproduct functor $\rightsquigarrow$ termination, exceptions, failure

Coalgebras

Transition systems are now called coalgebras:

Coalgebra

Let F be a given functor. A coalgebra is a function $\gamma\colon X \to FX$ (where X is the state set).

Metric transition systems

$$\gamma\colon X \to M \times \mathcal{P}_{fin}X$$

where M is a fixed metric space.

Probabilistic transition systems

$$\gamma\colon X \to \mathcal{D}X + 1$$

where 1 is a singleton set ($1 = \{\surd\}$), representing termination.

Coalgebras and Behavioural Metrics

Our (fibrational) approach: [Bonchi et al., 2023]

- Consider a **Set**-coalgebra $\gamma \colon X \to FX$.
- Lift functor on $F \colon \mathbf{Set} \to \mathbf{Set}$ to a functor on $\overline{F} \colon \mathbf{PMet} \to \mathbf{PMet}$ (transform metric on X to metric on FX)
- Obtain the behavioural metric on X as the least fixpoint of the following map f:

$$\begin{tikzcd} \mathbf{PMet}_X \arrow[r, "\overline{F}"] \arrow[rr, bend right, "f"'] & \mathbf{PMet}_{FX} \arrow[r, "\gamma^*"] & \mathbf{PMet}_X \end{tikzcd}$$

where $\mathbf{PMet}_X$ is the set of pseudo-metrics on X ("fibre" above X) and we use reindexing:

$$\gamma^*(d) = d \circ (\gamma \times \gamma)$$

Coalgebras and Behavioural Metrics

Needed: general methods for lifting a functor F to metric spaces

$\rightsquigarrow$ Wasserstein lifting, Kantorovich lifting

Evaluation functions and predicate liftings

We need a parameter: an evaluation function (algebra)

$$ev\colon F[0,1] \to [0,1]$$

Every evaluation function induces a real-valued predicate lifting:

$$(p\colon X \to [0,1]) \mapsto (ev \circ Fp\colon FX \to [0,1])$$

Notes:

- this can be extended to sets of evaluation maps.
- for the Wasserstein lifting, we have to put some requirements on the evaluation map.

Coalgebras and Behavioural Metrics

Wasserstein lifting

Let $d\colon X \times X \to [0,1]$ be a pseudo-metric and $t_1, t_2 \in FX$:

$$d^{\downarrow F}(t_1, t_2) = \inf\{ev(Fd(t)) \mid t \in F(X \times X), F\pi_i(t) = t_i\}$$

$$\begin{tikzcd} F(X \times X) \arrow[r, "\langle F\pi_1, F\pi_2 \rangle"] \arrow[rr, bend right, "ev \circ Fd"'] & FX \times FX \arrow[r, "d^{\downarrow F}"] & {[0,1]} \end{tikzcd}$$

- View d as a real-valued predicate and lift it.
- For each pair $(t_1, t_2) \in FX \times FX$ take a coupling in $F(X \times X)$ that gives the optimal (least) value (direct image).

Coalgebras and Behavioural Metrics

Kantorovich lifting (aka codensity lifting [Katsumata,Sato,'15])

Let $d\colon X \times X \to [0,1]$ be a pseudo-metric and $t_1, t_2 \in FX$:

$$d^{\uparrow F}(t_1,t_2) = \sup\{d_e(ev(Ff(t_1)), ev(Ff(t_2))) \mid f\colon (X,d) \to ([0,1],d_e) \text{ non-expansive}\}$$

where $d_e(x,y) = |x-y|$ for $x, y \in [0,1]$.

Given a pseudo-metric d:

- Take all non-expansive maps $f\colon (X,d) \to ([0,1],d_e)$ (right adjoint).
- Take the predicate lifting for each such map.
- Generate a pseudo-metric from the predicate liftings obtained in this way (least pseudo-metric that makes all predicate liftings non-expansive) (left adjoint).

Coalgebras and Behavioural Metrics

Results (Functor Lifting)

- $d^{\uparrow F}$, $d^{\downarrow F}$ are both pseudo-metrics (for the Wasserstein lifting we need some constraints on the evaluation function and weak pullback preservation)
- $d^{\uparrow F} \leq d^{\downarrow F}$
  There are cases where $d^{\uparrow F} < d^{\downarrow F}$, i.e., the Kantorovich-Rubinstein duality does not necessarily hold (e.g. for $FX = X \times X$).
- Non-expansive functions and isometries (distance-preserving functions) are preserved by lifting.
- The Wasserstein lifting preserves metrics (if the infimum is always a minimum).

Coalgebras and Behavioural Metrics

Several standard metrics can be recovered by lifting. In each of these cases the Kantorovich-Rubinstein duality holds.

functor	evaluation fct.	resulting metric
$\mathcal{P}_{fin}$	$ev(R \subseteq [0,1]) = \max R$	Hausdorff
$\mathcal{D}$	$ev(p\colon [0,1] \to [0,1]) = \sum_{x\in[0,1]} x \cdot p(x)$	Kantorovich
$X + Y$	$ev(x \in [0,1]) = x$	distance on disjoint union
$X \times Y$	$ev(x,y) = \max\{x,y\}$	maximum of distances
$X \times Y$	$ev(x,y) = x + y$	sum of distances

Last three cases: bifunctor lifting or use of multiple evaluation maps

Compositionality

Is functor lifting compositional?

Assume that $\overline{F}(X,d) = (FX, d^{\uparrow F})$ (Kantorovich) or $\overline{F}(X,d) = (FX, d^{\downarrow F})$ (Wasserstein)

When does $\overline{FG} = \overline{F}\,\overline{G}$ hold?

- Wasserstein:
  - $\overline{F}\,\overline{G} \le \overline{FG}$ always holds (if evaluation maps are composed)
  - Equality for so-called canonical liftings
- Kantorovich:
  - $\overline{FG} \le \overline{F}\,\overline{G}$ always holds (if evaluation maps are composed)
  - Equality if F polynomial and chosen evaluation maps

Useful for defining up-to techniques that help to simplify (co)inductive proofs . . .

Up-To Techniques

Let $f\colon L \to L$ be a monotone function on a complete lattice. By Knaster-Tarski this function has a least fixpoint μf (that coincides with the least pre-fixpoint) and a greatest fixpoint νf (that coincides with the greatest post-fixpoint).

Idea: extend the usual proof rule for pre-fixpoints to a proof rule using an up-to function u.

$$\frac{f(d) \le d}{\mu f \le d} \qquad \frac{f(u(d)) \le d}{\mu f \le d}$$

Bialgebras: Combining Algebras and Coalgebras

Add an algebraic structure that interacts "nicely" with the coalgebraic structure:

Bialgebra

Consider two functors $F, T\colon \mathbf{Set} \to \mathbf{Set}$ and a distributive law for a monad $\zeta\colon TF \Rightarrow FT$.

A bialgebra for ζ consists of a T-algebra $\beta\colon TX \to X$ and an F-coalgebra $\gamma\colon X \to FX$ so that the diagram commutes.

$$\begin{tikzcd}
TX \arrow[r, "\beta"] \arrow[d, "T\gamma"'] & X \arrow[r, "\gamma"] & FX \\
TFX \arrow[rr, "\zeta_X"] & & FTX \arrow[u, "F\beta"']
\end{tikzcd}$$

Such bialgebras can be obtained by determinizing coalgebras of the form $Y \to FTY$. In this case $X = TY$.

Up-To Techniques

Coinduction Up-To [Bonchi,Petrișan,Pous,Rot, 2017]

Given a bialgebra $\beta\colon TX \to X$, $\gamma\colon X \to FX$ (with distr. law ζ):

- Lift the functors F, T to $\overline{F}, \overline{T}\colon \mathbf{PMet} \to \mathbf{PMet}$
- This gives us an up-to function u on the fibre above $\mathbf{PMet}_X$:

$$\begin{tikzcd} \mathbf{PMet}_X \arrow[r, "\overline{T}"] \arrow[rr, bend right, "u"'] & \mathbf{PMet}_{TX} \arrow[r, "\Sigma_\beta"] & \mathbf{PMet}_X \end{tikzcd}$$

where $\Sigma_\beta(d)(x_1, x_2) = \inf_{\beta(t_i)=x_i} d(t_1, t_2)$ (direct image).
u is typically some form of metric congruence closure wrt. the operators of the algebra.

Up-To Techniques

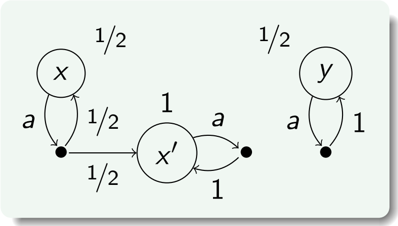
- Whenever ζ can be extended to a natural transformation $\zeta\colon \overline{T}\,\overline{F} \Rightarrow \overline{F}\,\overline{T}$ (i.e., the components of ζ are non-expansive maps), the following proof rule holds:

$$\frac{f(u(d)) \leq d}{\mu f \leq d}$$

Up-to functions help to find witnesses (pre-fixed-points up-to), establishing upper bounds for least fixpoints.

Typically it is easier to show $f(u(d)) \leq d$ rather than $f(d) \leq d$.

Behavioural Metrics for Probabilistic Automata



$Q \to [0,1] \times \mathcal{D}(Q)^A$

$tr_x(w)$	ε	a	aa	aaa	...
x	$1/2$	$3/4$	$7/8$	$15/16$	...
y	$1/2$	$1/2$	$1/2$	$1/2$	...

For a state $x \in Q$ let $tr_x \colon \Sigma^* \to [0,1]$ assign to each word $w \in \Sigma^*$ the expected payoff for this word when read from x.

Directed behavioural distance between states:

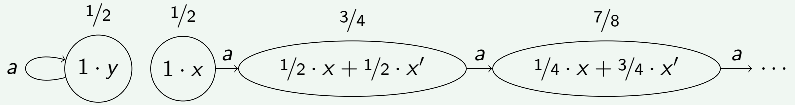
$$d(x,y) = \sup_{w \in \Sigma^*} (tr_y(w) \ominus tr_x(w))$$

In this case: $d(x,y) = 1/2$.

This can be extended to probability distributions on states.

Behavioural Metrics for Probabilistic Automata

Probabilistic determinization:



Behavioral distance can be characterized as least fixpoint of f:

$$\begin{aligned} f\colon [0,1]^{\mathcal{D}(Q)\times\mathcal{D}(Q)} &\to [0,1]^{\mathcal{D}(Q)\times\mathcal{D}(Q)} \\ f(d)(p,q) &= \max\{|tr_p(\varepsilon) - tr_q(\varepsilon)|, \\ &\qquad \max_{a\in\Sigma}\{\delta_a(p), \delta_a(q)\}\} \end{aligned}$$

$tr_p(\varepsilon)$: immediate payoff at p

Unfortunately, the state space is infinite . . .

Behavioural Metrics for Probabilistic Automata

Aim: show that $d(x,y) \leq 1/2$ in the example above.

Define finitary witness $\bar{d}$:

- $\bar{d}(1 \cdot x, 1 \cdot y) = 1/2$, $\bar{d}(1 \cdot x', 1 \cdot y) = 1/2$ (1 in all other cases)

and show that $\bar{d}$ is a pre-fixpoint up-to.

$$\begin{aligned} f(u(\bar{d}))(1 \cdot x, 1 \cdot y) &= \max\{|1/2 - 1/2|, u(\bar{d})(1/2 \cdot x + 1/2 \cdot x', y)\} \\ &= u(\bar{d})(1/2 \cdot x + 1/2 \cdot x', y) \\ &\leq 1/2 \cdot \bar{d}(1 \cdot x, 1 \cdot y) + 1/2 \cdot \bar{d}(1 \cdot x', 1 \cdot y) \\ &= 1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/2 = \bar{d}(1 \cdot x, 1 \cdot y) \qquad \checkmark ? \end{aligned}$$

The other case works similarly $\Rightarrow \mu f = d \leq \bar{d}$.

Use up-to approach to show that this reasoning is sound!

Behavioural Metrics for Probabilistic Automata

Instantiating the general approach:

- $T = \mathcal{D}$
- $FX = [0,1] \times X^\Sigma$
- State space $X = \mathcal{D}(Q)$ (probability distributions on Q)
- Coalgebra $\gamma\colon X \to FX$:
  $\gamma\colon \mathcal{D}(Q) \to [0,1] \times \mathcal{D}(Q)^\Sigma$ (determinized probabilistic automaton)
- Algebra $\beta\colon TX \to X$:
  $\beta\colon \mathcal{D}(\mathcal{D}(D)) \to \mathcal{D}(Q)$ (expectation/"flattening" nested probability distributions; multiplication of the monad)
- $\zeta_X\colon \mathcal{D}([0,1] \times X^\Sigma) \to [0,1] \times \mathcal{D}(X)^\Sigma$ ("probabilistic determinization")

Behavioural Metrics for Probabilistic Automata

Lifting the functors $F, T\colon \mathbf{Set} \to \mathbf{Set}$ to $[0,1]$-**Graph**:
Let $d\colon X \times X \to [0,1]$.

- $FX = [0,1] \times X^{\Sigma}$:

$$\overline{F}(d)((a,x),(b,y)) = \max\{d_{\Sigma}(a,b), d(x,y)\}$$

  (where d_{Σ} is a fixed metric on Σ).

- $T = \mathcal{D}$:
  $\overline{T}(d)$ is the Kantorovich distance on probability distributions

Behavioural Metrics for Probabilistic Automata

- $f = \gamma^* \circ \overline{F}$: behavioural distance function on the determinized probabilistic transition system ($\overline{F}$: Kantorovich lifting)
- $u = \Sigma_\beta \circ \overline{T}$: contextual closure for barycentric algebras

  $$u(d)(r_1 \cdot p_1 + r_2 \cdot p_2, r_1 \cdot q_1 + r_2 \cdot q_2) \leq r_1 \cdot d(p_1, q_1) + r_2 \cdot d(p_2, q_2)$$

  See also work on quantitative algebraic reasoning [Mardare,Panangaden,Plotkin '16]

We proved some simple conditions ensuring that the distributive law can be lifted (for F polynomial and $\overline{T}$ Kantorovich lifting) (plus additional results on Galois connections and compositionality): [D'Angelo,Gurke,Kirss,König,Najafi,Różowski,Wild, 2024]

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