Syntax and semantics for mechanical processes

Topos Institute Colloquium

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Collaborators

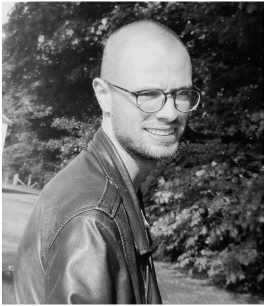
Aleks Kissinger

University of Oxford

- “A Graphical Calculus for Lagrangian Relations”
  `arXiv:2105.06244`

Collaborators

Robert I. Booth



University of Oxford

Titouan Carette

LIX - École polytechnique

- "Graphical Symplectic Algebra"
  `arXiv:2401.07914`
- "Complete equational theories for classical and quantum Gaussian relations"
  `arXiv:2403.10479`

Collaborators

Giovanni de Felice

Quantinuum

- “The delayed stabilizer ZX-calculus”
  forthcoming

Motivation

1. Find an interesting problem in physics.
2. Formalize it categorically.
3. Give a presentation in terms of generators and equations.

Class of mechanical circuits
(physical semantics)

Category of relations
(Categorical semantics)

string diagrams+equations
(graphical syntax)

I like creating graphical languages, physics gives me examples!

Outline

1. Review linear algebra using string diagrams.
2. Sketch the basic idea of Hamiltonian mechanics using symplectic geometry.
3. Combining the previous two points, we give graphical languages for various classes of mechanical processes.

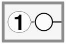
Affine matrices

Definition

The symmetric monoidal category AffMat_k has:

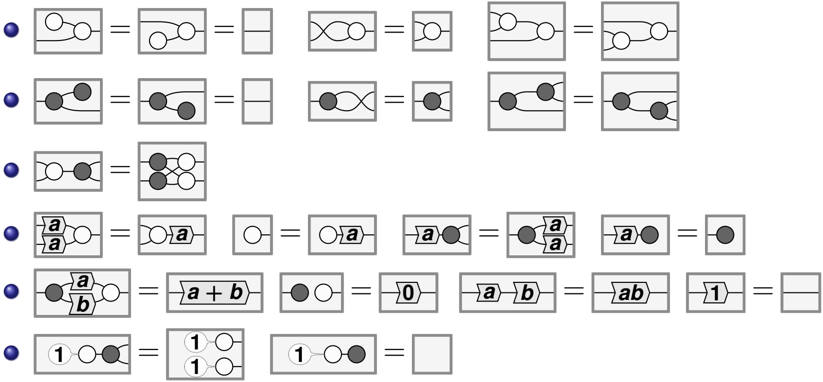
- **Objects:** natural numbers;
- **Morphisms:** $(T \in k^{m \times n}, \mathbf{a} \in k^m) : n \to m$;
- **Identity:** $1_n := (I_n, \mathbf{0}) : n \to n$;
- **Composition:** $\dfrac{(T, \mathbf{a}) : n \to m, \quad (S, \mathbf{b}) : m \to k}{(S, \mathbf{b}) \circ (T, \mathbf{a}) := (S \circ T, S \circ \mathbf{a} + \mathbf{b}) : n \to \ell}$
- **Monoidal product:**
$$\frac{(T, \mathbf{a}) : n \to m, \quad (S, \mathbf{b}) : \ell \to r}{(T, \mathbf{a}) \otimes (S, \mathbf{b}) := \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}, \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \right) : n + m \to \ell + r}$$
- **Monoidal unit:** $I := 0$.

AffMat_k is generated by, for all $a \in k$:

- $[\![$  $]\!] = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0 \right) : 1 \to 2;$
- $[\![$  $]\!] = (0, 0) : 1 \to 0;$
- $[\![$  $]\!] = \left(\begin{bmatrix} 1 & 1 \end{bmatrix}, 0 \right) : 2 \to 1;$
- $[\![$  $]\!] = (0, 0) : 0 \to 1;$
- $[\![$  $]\!] = (a, 0) : 1 \to 1;$
- $[\![$  $]\!] = (0, 1) : 0 \to 1;$

Equations for affine matrices (folklore)

Modulo the equations, for all $a, b \in k$:



Affine relations

Definition

The symmetric monoidal category AffRel_k has:

- **Objects**: natural numbers;
- **Morphisms**: $n \to m$ are affine subspaces $L + \mathbf{a} = \{\mathbf{v} - \mathbf{a} \mid \forall v \in L\} \subseteq k^n \oplus k^m$ or empty $\varnothing \subseteq k^n \oplus k^m$;
- **Identity**: $1_n = \{(\mathbf{v}, \mathbf{v}) \mid \forall \mathbf{v} \in k^n\}$;
- **Composition**:
$$\frac{R : n \to m, \quad S : m \to \ell}{S \circ R := \{(\mathbf{v}, \mathbf{w}) \mid \exists \mathbf{u} \in k^m : (\mathbf{v}, \mathbf{u}) \in R, (\mathbf{u}, \mathbf{w}) \in S\} : n \to \ell};$$
- **Monoidal product**: direct sum;
- **Monoidal unit**: 0.

From affine transformations to affine relations

The graph of an affine transformation is a symmetric monoidal functor $\mathsf{Gr}^* : \mathsf{AffMat}_k \rightarrowtail \mathsf{AffRel}_k$

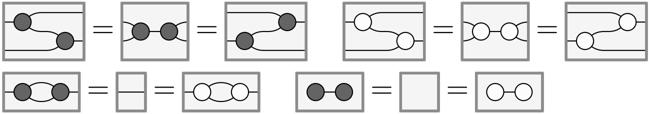
$$(A : n \to m) \quad \longmapsto \quad (\{(\mathbf{v}, A \circ \mathbf{v}) \mid \forall \mathbf{v} \in k^n\} : n \to m)$$

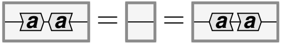
Similarly, the cograph of an affine transformation is a symmetric monoidal functor $\mathsf{Gr}_* : \mathsf{AffMat}_k^{\mathrm{op}} \rightarrowtail \mathsf{AffRel}_k$

$$(A : n \to m) \quad \longmapsto \quad (\{(A \circ \mathbf{v}, \mathbf{v}) \mid \forall \mathbf{v} \in k^n\} : m \to n)$$

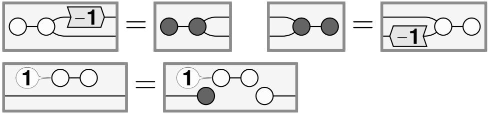
Presentation for affine relations (Bonchi et al. 2019)

AffRel_k is generated by the the generators and equations of $\mathsf{Gr}^*(\mathsf{AffMat}_k)$ and $\mathsf{Gr}_*(\mathsf{AffMat}_k)$ in addition to the equations:

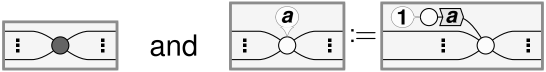




for all $a \in k$: $a \neq 0$



White/grey “spiders” denote connected components of white/grey Frobenius algebras:



How do we connect this to physics?

1. We want to refine $\mathsf{AffMat}_{\mathbb{R}}$ so that morphisms are the equations of motion of particles.
2. We want to refine $\mathsf{AffRel}_{\mathbb{R}}$ so that morphisms are the affinely constrained flows of particles.
3. Then we will give a more refined graphical language...

Phase space

In Hamiltonian mechanics, the **position space** is represented by a manifold M.

The **phase space** is the configuration space of position and *momentum*; represented by the cotangent bundle T^*M.

We interpret a particle as a point $(q, p) \in T^*M$.

Example

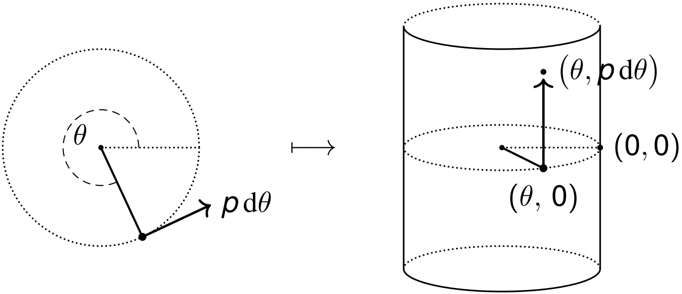
For translational momentum: $T^*\mathbb{R}^n \cong \mathbb{R}^n \times (\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$

Visualizing angular momentum

For this talk, we care about position space $M = \mathbb{R}^n$, but the circle is useful to build intuition:

Example

For angular momentum: $T^*\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$



Generalized positions and momenta (Baez and Fong 2018)

In different mechanical settings, there are different notions of position and momentum:

Classical mechanics	q	dq/dt	p	dp/dt
Translation	position	velocity	momentum	force
Rotational	angle	angular velocity	angular momentum	torque
Electronic	charge	current	"flux linkage"	voltage
Hydraulic	volume	flow	"pressure momentum"	pressure
Thermal	entropy	"entropy flow"	"temperature momentum"	temperature

Symplectic structure

In general T^*M is a *symplectic manifold*. More specifically, $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ is a symplectic $\mathbb{R}$-vector space:

Definition

A **symplectic k-vector space** is a pair (V, ω_V) where:

- V is a k-vector space;
- $\omega_V : V \times V \to k$ is a non-degenerate, alternating bilinear form.

Similarly, a k-**linear symplectomorphism** $(V, \omega_V) \to (W, \omega_W)$ is a k-linear isomorphism $T : V \to W$ such that for all $\mathbf{a}, \mathbf{b} \in V$:

$$\omega_W(T \circ \mathbf{a}, T \circ \mathbf{b}) = \omega_V(\mathbf{a}, \mathbf{b})$$

Lemma (Darboux)

Every finite-dimensional symplectic k-vector space (V, ω_V) is symplectomorphic to (k^{2n}, ω_n) where:

$$\omega_n((q,p),(q',p')) := q^\top \circ p' - p^\top \circ q'$$

Hamiltonian mechanics

Symplectic geometry is the mathematics of *Hamiltonian mechanics*.

- An (autonomous) Hamiltonian is a smooth function $H : T^*M \to \mathbb{R}$ assigning energy values to particles.
- This induces a gradient $dH : TT^*M \to \mathbb{R}$.
- There is a vector bundle isomorphism $\Phi : TT^*M \to T^*T^*M$ given by $\Phi(v) := \omega(v, -)$.
- These define a Hamiltonian vector field $\mathfrak{X}_H := \Phi^{-1}(dH) : T^*M \to TT^*M$.

- **The 1-step Hamiltonian evolution** $\exp(\mathfrak{X}_H) : T^*M \to T^*M$ **is a symplectomorphism!**
  $\Rightarrow$ symplectomorphisms generalize equations of motion.

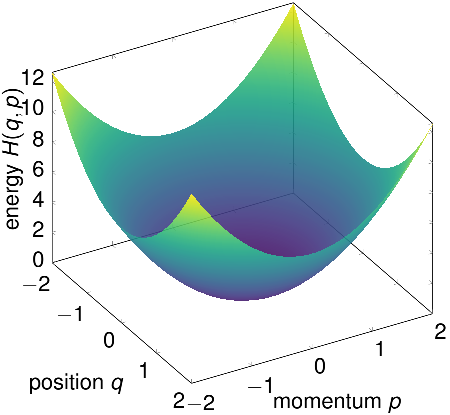


Figure: Hamiltonian $H(q, p) = \frac{\theta}{2}\left(q^2 + p^2\right)$ for an oscillating spring.

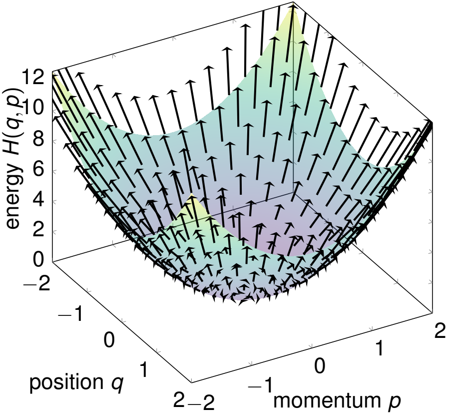


Figure: $dH = \theta q\, dq + \theta p\, dp$ drawn on the surface of $H(q,p) = \frac{\theta}{2}(q^2 + p^2)$.

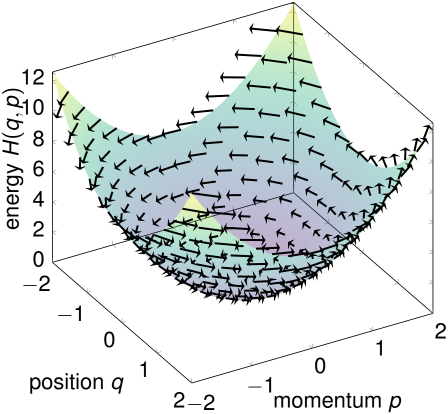


Figure: Hamiltonian vector field $\mathfrak{X}_H = \theta p \frac{\partial}{\partial q} - \theta q \frac{\partial}{\partial p}$ drawn on the surface $H(q,p) = \frac{\theta}{2}(q^2 + p^2)$.

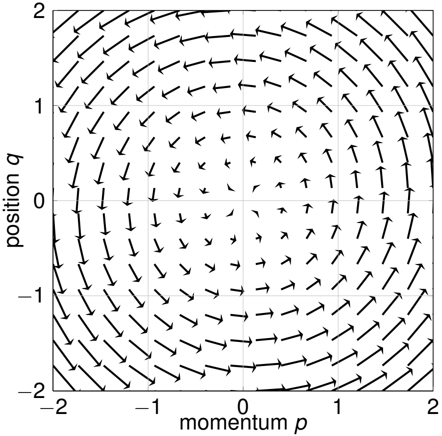


Figure: Hamiltonian vector field $\mathfrak{X}_H = \theta p \frac{\partial}{\partial q} - \theta q \frac{\partial}{\partial p}$ for $H(q,p) = \frac{\theta}{2}(q^2 + p^2)$. Exponentiating the vector field $\exp\left(\theta p \frac{\partial}{\partial q} - \theta q \frac{\partial}{\partial p}\right)$ rotates by θ.

Quadratic Hamiltonians

For our example of $M = \mathbb{R}^n$ and $T^*(\mathbb{R}^n) \cong \mathbb{R}^{2n}$:

Lemma

Given a quadratic Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ *so that*
$\mathbf{v} \mapsto \frac{1}{2}\mathbf{v}^{\mathsf{T}} \circ A \circ \mathbf{v} + \mathbf{b}^{\top} \circ \mathbf{v} + c$ *for* $A \in \mathrm{Sym}_{2n}(\mathbb{R}), \mathbf{b} \in \mathbb{R}^{2n}, c \in \mathbb{R}$
the Hamiltonian evolution is an affine symplectomorphism:

$$\left(\exp(\Omega_n \circ A), \left(\int_0^1 \exp((1-s)\Omega_n \circ A) ds\right) \circ (\Omega_n \circ \mathbf{b})\right) : (\mathbb{R}^{2n}, \omega_n) \to (\mathbb{R}^{2n}, \omega_n)$$

Where $\Omega_n := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ *so that* $\omega_n(\mathbf{v}, \mathbf{w}) = \mathbf{v}^{\top} \circ \Omega_n \circ \mathbf{w}$.

Example

Given the quadratic Hamiltonian $\mathbb{R}^2 \to \mathbb{R}$ such that

$$(q, p) \mapsto \frac{\theta}{2}\left((q-a)^2 + (p-b)^2\right)$$

The Hamiltonian evolution is the affine symplectomorphism $(\mathbb{R}^2, \omega_1) \to (\mathbb{R}^2, \omega_1)$ which rotates about (a, b) by θ.

Affine Lagrangian subspaces

Definition

Given a linear subspace $S \subset (V, \omega_V)$ of a symplectic vector space, the **symplectic complement** is the linear subspace:

$$S^{\omega_V} := \{\mathbf{v} \in V \mid \forall \mathbf{s} \in S : \omega_V(\mathbf{v}, \mathbf{s}) = 0\}$$

An **affine Lagrangian subspace** $S \subset (V, \omega_V)$ is an affine subspace $S + \mathbf{a} \subseteq V$ such that $S^{\omega_V} = S$.

Affine Lagrangian subspaces generalize affine symplectomorphisms, and thus, the equations of motion given by quadratic Hamiltonians:

Lemma

Given an affine symplectomorphism $S : (V, \omega_V) \to (W, \omega_W)$, its graph is an affine Lagrangian subspace

$$\mathrm{Gr}^*(S) \subseteq (V \oplus W, -\omega_V \oplus \omega_W)$$

Affine Lagrangian relations

Definition (Weinstein)

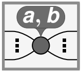
The symmetric monoidal category $\mathsf{AffLagRel}_k$ has:

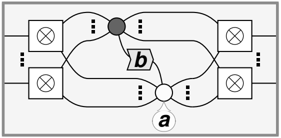
- **Objects:** Natural numbers;
- **Morphisms:** $n \to m$ given by affine Lagrangian subspaces of $(k^{2n} \oplus k^{2m}, -\omega_n \oplus \omega_m)$ or the empty set;
- **Composition, identity and monoidal structure:** same as in AffRel_k

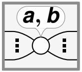
Affine symplectomorphisms are generalized equations of motion.

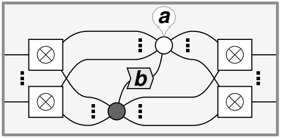
Affine Lagrangian relations are nondeterministic generalizations of equations of motion: morphisms $n \to m$ are affine constraints dictating how n particles can *flow* into m particles.

Affine Lagrangian relations is generated by (a, b)-labelled spiders, for all $a, b \in k$:

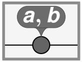
$[\![$ 

$]\!] :=$ 

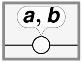
$[\![$ 

$]\!] :=$ 

So that in particular:

$[\![$ 

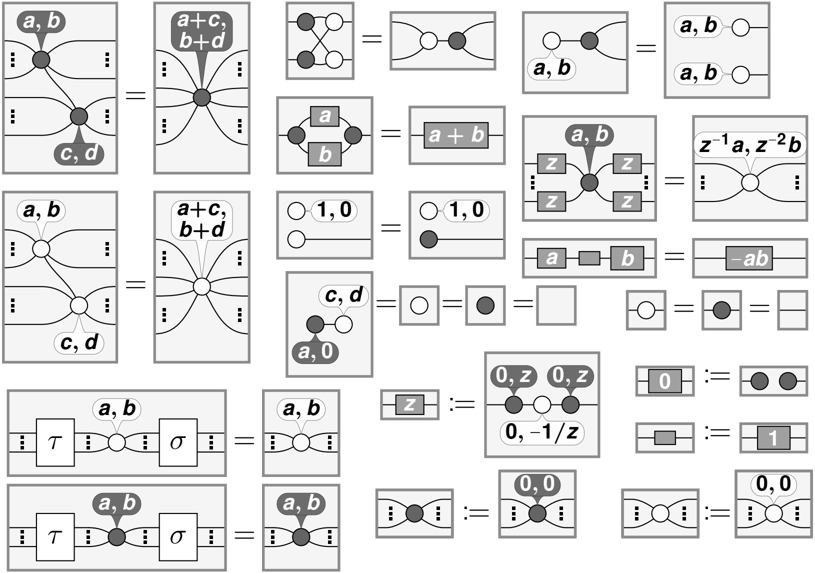
$]\!] = \mathrm{Gr}^*\left(\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ a \end{bmatrix}\right) : 1 \to 1$

$[\![$ 

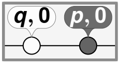
$]\!] = \mathrm{Gr}^*\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a \\ 0 \end{bmatrix}\right) : 1 \to 1$

Equations (Booth, Carette, Comfort, 2024)

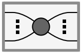
For all $a, b, c, d, z \in k$ such that $z \neq 0$ and permutations τ, σ:



Example: Electrical circuits

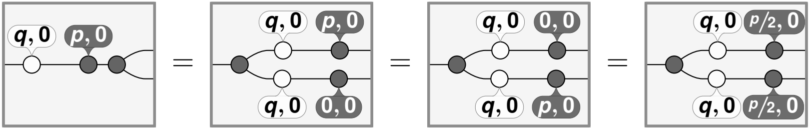


represents a particle $(q, p) \in \mathbb{R}^2$.
Recall q is charge, ${}^{dq}/_{dt}$ is current and ${}^{dp}/_{dq}$ is voltage.

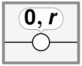

$: n \to m$ represents an (idealized) junctions of wires.

"Voltages across wires are equal; charge is conserved."

Ex: there are various possible ways particles can flow:



Example: Electrical circuits

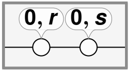


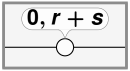
$: 1 \to 1$ is a symplectomorphism with Hamiltonian

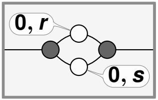
$$(q, p) \mapsto \tfrac{r}{2} q^2.$$

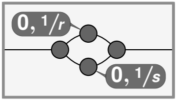
This is interpreted as a non-dissipative (idealized) resistor with resistance $r \in \mathbb{R}^+$.

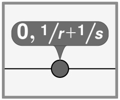
We can compose resistors:

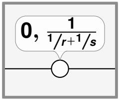
- **in sequence**: 

$=$ 

- **in parallel:** 

$=$ 

$=$ 

$=$ 

Example: Stabilizer quantum circuits

By replacing $\mathbb{R}$ with $\mathbb{F}_p$ for odd prime p, there is an embedding: $\mathsf{AffLagRel}_{\mathbb{F}_p} \hookrightarrow (\mathsf{FHilb}, \otimes, \mathbb{C})/\sim$ modulo nonzero scalars.

- **On objects:** $n \mapsto \mathbb{C}[\mathbb{F}_p^n] := \mathrm{Span}_{\mathbb{C}}\{|\mathbf{v}\rangle \mid \forall \mathbf{v} \in \mathbb{F}_p^n\}$;
- **States:** $0 \to n$ are sent to “stabilizer states;”
- **Morphisms** $n \to m$ are sent to “stabilizer circuits;”
- **Isomorphisms** $n \to m$ are sent to “Clifford operators.”

This is compatible with the classical picture of Hamiltonian mechanics:

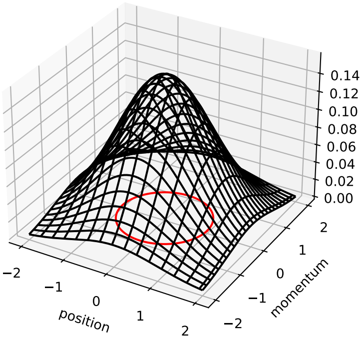
$$\begin{tikzcd}
 & \begin{array}{c}\text{quadratic forms}\\ \mathbb{F}_p^{2n} \to \mathbb{F}_p\end{array} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\begin{array}{c}\text{Clifford operators}\\ \mathbb{C}[\mathbb{F}_p^n] \to \mathbb{C}[\mathbb{F}_p^n]\end{array} \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{Affine Symplectomorphisms}\\ (\mathbb{F}_p^{2n}, \omega_n) \to (\mathbb{F}_p^{2n}, \omega_n)\end{array}
\end{tikzcd}$$

Stabilizer quantum circuits are extremely important in quantum error correction.

In statistical mechanics, or continuous-variable quantum mechanics, particles are:

probability distributions

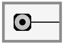
The ground state is interpreted as the standard Gaussian distribution in phase space $T^*(\mathbb{R}) \cong \mathbb{R}^2$:

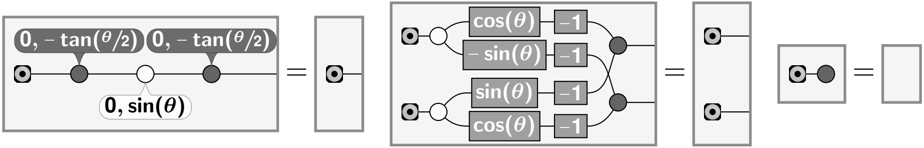


Gaussian mechanics (Booth, Carette, Comfort, 2024)

Semantically, Gaussian mechanics can be interpreted in the subcategory $\mathsf{AffLagRel}_{\mathbb{C}}$, imposing that $\forall \mathbf{v} \in L + \mathbf{a},\ i\omega(\mathbf{v}, \overline{\mathbf{v}}) \geq 0$.

This is generated by the two spiders of $\mathsf{AffLagRel}_{\mathbb{C}}$ where the labels are restricted to $\{(a, b + ci) \in \mathbb{C}^2 \mid a, b, c \in \mathbb{R}, c \geq 0\}$.

Syntactically, this is represented by the equational theory of $\mathsf{AffLagRel}_{\mathbb{R}}$, in addition to a generator  for the ground state which is invariant under rotations:



Adding a time delay (Comfort, de Felice, forthcoming)

We can model time-dependent mechanical systems by a time-delay generator $\boxed{\delta}$ to the syntax of $\mathrm{AffLagRel}_k$.
By working with affine relations over the rational functions

$$k(\delta) := \left\{ \frac{f(\delta)}{g(\delta)} \;\middle|\; \forall f(\delta), g(\delta) \in k[\delta] : g(\delta) \neq 0 \right\}$$

We can interpret the delay as multiplication by δ:

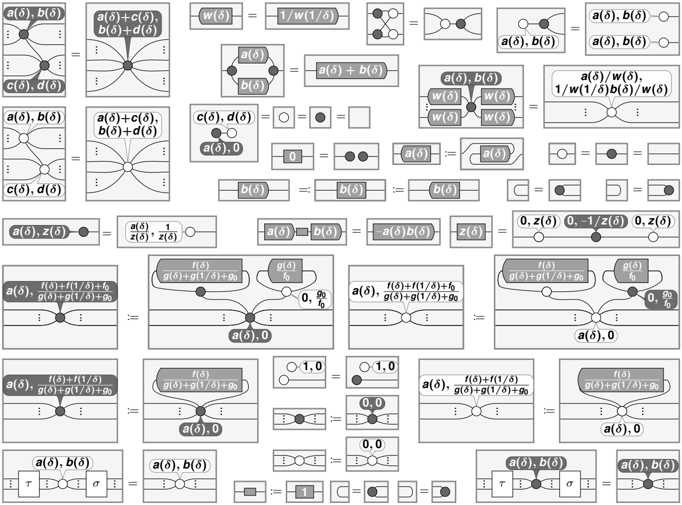
$$\left[\!\left[\boxed{\delta} \right]\!\right] := \left\{ \left(\begin{bmatrix} q(\delta) \\ p(\delta) \end{bmatrix}, \begin{bmatrix} q(\delta)\cdot\delta \\ p(\delta)\cdot\delta \end{bmatrix} \right) \;\middle|\; \forall q(\delta), p(\delta) \in k(\delta) \right\}$$

$\boxed{\delta}$ + $\mathrm{AffLagRel}_k$ generates *shifted* affine Lagrangian relations, where the symplectic form is twisted with a conjugation:

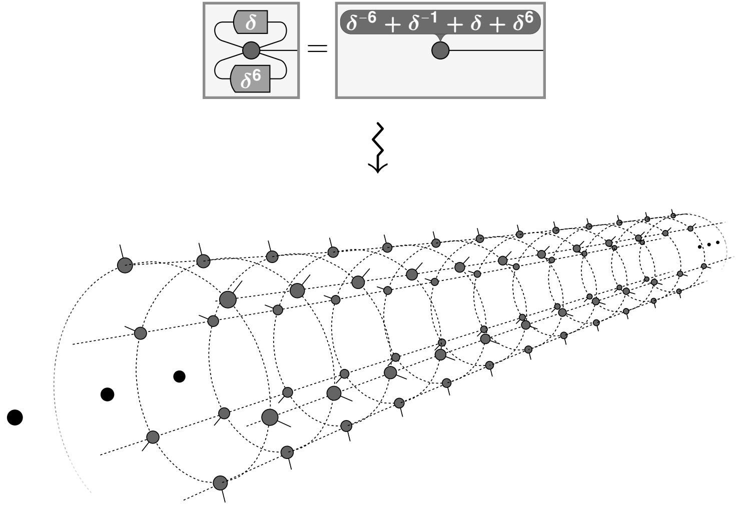
$$\widetilde{\omega}_n(f(\delta), g(\delta)) := \omega_n(f(\delta), g(1/\delta))$$

Axioms of the delayed stabiliser ZX-calculus

For all permutations σ and τ, $a(\delta), c(\delta), w(\delta) \in k(\delta)$, $b(\delta), d(\delta), z(\delta) \in k(\delta + 1/\delta)$, $f(\delta), g(\delta) \in k[\delta]$, and $f_0, g_0 \in k$ such that $f(0) = g(0) = 0$, $g(\delta) + g(1/\delta) + g_0 \neq 0$, $f_0 \neq 0$, $w(\delta) \neq 0$, and $z(\delta) \neq 0$.



Cool pictures involved



Thank you

Questions?