Lax distributivity of (co)lax algebras

$$\begin{tikzcd} \mathcal{C} \arrow[r, "m"] \arrow[d, "c"'] \arrow[dr, phantom, "\nearrow\Downarrow"] & \mathcal{C} \arrow[d, "c"] \\ \mathcal{C} \arrow[r, "m"'] & \mathcal{C} \end{tikzcd}$$

λ interacts with the (co)monad structures

$\mathcal{J}, \mathcal{K}$ small cats $\quad D : (\mathcal{C}^{\mathcal{J}})^{\mathcal{K}} \to \mathcal{C}$

$$\lim_{\mathcal{J}} (\lim_{\mathcal{K}} D) \simeq \lim_{\mathcal{K}} (\lim_{\mathcal{J}} D)$$

$$\operatorname{colim}_{\mathcal{J}} (\lim_{\mathcal{K}} D) \xrightarrow{\exists!} \lim_{\mathcal{J}} (\operatorname{colim}_{\mathcal{K}} D)$$

Abelian category (additive + ker + coker)

$$\ker f \overset{\iota}{\rightarrowtail} a \xrightarrow{f} b \overset{\pi}{\twoheadrightarrow} \operatorname{coker} f$$

$$\operatorname{coim} f = \operatorname{coker}(\ker f \to a) \xrightarrow[\exists!]{\cong} \ker(b \xrightarrow{\pi} \operatorname{coker} f) = \operatorname{im} f$$

Monoidal monad

$(\mathcal{C}, \otimes, I)$, $(\mathcal{C} \xrightarrow{m} \mathcal{C}, \mu, \eta)$

$ma \otimes mb \to m(a \otimes b)$ + satisfy compatibility

What does these have in common?

(T_1, μ_1, η_1) a 2-monad
(T_2, μ_2, η_2) — " —
$\Big\}\ \lambda : T_1T_2 \Rightarrow T_2T_1$

(co)lax $T_1A \xrightarrow{a_1} A$, (colax) $T_2A \xrightarrow{a_2} A$

$$\begin{tikzcd} T_1T_2A \arrow[r, "T_1a_2"] \arrow[d] \arrow[dr, phantom, "\xi\Downarrow"] & T_1A \arrow[d, "a_1"] \\ T_2T_1A \arrow[d, "T_2a_1"'] & \\ T_2A \arrow[r, "a_2"] & A \end{tikzcd}$$

+ coherence

Introduced M. Markl + T. : lax distr. λ invertible, normalized algebras

& a characterization of Ab. cats

Ongoing: w. M. Štěpán. ← non invertible, unknown.

Idea clear ... good definition?

Classical case (J. Beck 1969)

Two monads (T_i, μ_i, η_i) $i = 1, 2$ $\circlearrowleft \mathcal{C}$

$\lambda : T_1T_2 \Rightarrow T_2T_1$ + compatibility of λ w. μ_i and η_i

- Composite monad $T_{21} = T_2T_1$

  $T_{21}e = T_2T_1e$, $\quad T_2T_1T_2T_1 \xRightarrow{T_2\lambda T_1} T_2T_2T_1T_1 \xRightarrow{\mu_2\mu_1} T_2T_1$

- lift the monad T_2 to T_1-algebras:

$$T_1T_2 \circlearrowright \mathcal{C} \leftrightarrows \mathcal{C}^{T_1} \circlearrowleft \tilde{T}_2$$

$$\tilde{T}_2(A, T_1A \xrightarrow{a_1} A) = (T_2A, T_1T_2A \xrightarrow{\lambda_A} T_2T_1A \xrightarrow{T_2a_1} T_1A)$$

λ is invertible

Def: Rewriting rule:

$T_1A \xrightarrow{a_1} A$, $T_2A \xrightarrow{a_2} A$ satisfy rewr rule if

$$\begin{tikzcd} T_1T_2A \arrow[r, "T_1a_2"] \arrow[d, leftrightarrow, "\lambda"] & T_1A \arrow[d, "a_1"] \\ T_2T_1A \arrow[r, equal] \arrow[d, "T_2a_1"'] & T_1A \arrow[d, "a_1"] \\ T_2A \arrow[r, "a_2"] & A \end{tikzcd}$$

Proposition: TFAE (invertible)

1) pair of T_1-alg and T_2-alg sat. r.r.
2) pair of T_1-alg and T_2-alg are s.t. a_2 is a morphism of T_1-alg's + a_1 a morph of T_2-alg's
3) $\tilde{T}_2$-alg $((A, a_1), a_2)$
4) $\tilde{T}_1$-alg $((A, a_2), a_1)$
5) T_{21}-alg $(A, a_2 \cdot T_2a_1)$

The lax world

(T, μ, η) 2-monad

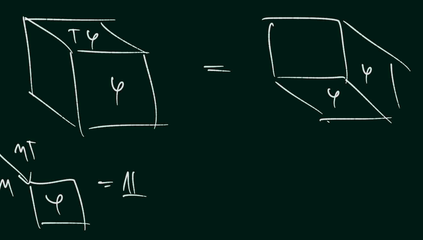
colax T-algebra is (normalized)

A, $TA \xrightarrow{a} A$,

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\varphi\Downarrow"] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

+ coherence

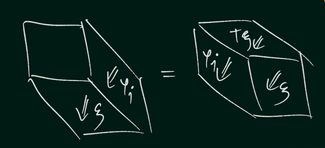
- $A \xrightarrow{\eta} TA \xrightarrow{a} A$, $\eta a = 1_A$



Def a lax distributivity of a (mixed case) colax T_1-alg (A, a_1, φ_1) and lax T_2-alg (A, a_2, φ_2) is a 2-cell ξ

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\lambda"'] \arrow[dr, phantom, "\xi\Downarrow"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

+ coherence:



similar but a bit different

($\forall$ unit axioms)

Proposition: Let (A, a_1, φ_1) be a colax T_1-alg; (A, a_2, φ_2) lax T_2-alg $(T_1T_2 \overset{\lambda}{\Rightarrow} T_2T_1)$

let ξ be 2-cell

TFAE

1) ξ is a lax distributive law
2) colax morph. $\tilde{T}_1(A, a_2, \varphi_2) \xrightarrow{(a_1, \xi)} (A, a_2, \varphi_2)$ of lax alg's

   & lax morph $\tilde{T}_2(A, a_1, \varphi_1) \xrightarrow{(a_2, \xi)} (A, a_1, \varphi_1)$ of colax alg's

   (colax T_1-alg, lax $\tilde{T}_2$)
3) lax $\tilde{T}_2$-alg on (A, a_1, φ_1) (colax) ✓
4) colax $\hat{T}_1$-alg on (A, a_2, φ_2) (lax)

(5) not that clear)

Examples:

- $\mathbb{1}_{Cat}$ — lax $\mathbb{1}$-alg $\simeq$ monads $(\mathcal{C} \xrightarrow{a} \mathcal{C}, \varphi, \iota)$; $(\mathcal{C}, a, \varphi, \iota)$ colax — comonads
- M — pseudo M-alg $\approx$ monoidal categories
- $\mathcal{J}$ small cat. $\quad 1 \xleftarrow{!} \mathcal{J} \xrightarrow{\Delta} \mathcal{J} \times \mathcal{J}$

  $\square^{\mathcal{J}} : \mathcal{C} \mapsto \mathcal{C}^{\mathcal{J}}$, $\quad \mathcal{C}^{\mathcal{J}} \underset{\text{const}}{\overset{\text{colim}}{\rightleftarrows}} \mathcal{C}$, $\quad T \dashv$

  special (co)lax algebra

- $\mathcal{J} = \{0 \to 1\} = 2$, $\quad \square^2$ squaring monad

  pseudo alg $\square^2$-alg = orthogonal fact. sys's.

- $\circlearrowleft \square^2$ PtCat (different structure!) lax alg's are are pt. cats with ker's cokernels

  colax $\rightsquigarrow$

- $\mathcal{F}\mathcal{G}\mathrm{Cat}/\mathcal{C}$ — pseudoalg's are fibrations over $\mathcal{C}$

$T_1 = T_2 = \mathbb{1}_{Cat}$, $\lambda : \mathbb{1} \overset{\mathbb{1}}{\Rightarrow} \mathbb{1}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "m"] \arrow[d, "c"'] \arrow[dr, phantom, "\xi\Downarrow"] & \mathcal{C} \arrow[d, "c"] \\ \mathcal{C} \arrow[r, "m"'] & \mathcal{C} \end{tikzcd}$$

ξ — compatibility $\Updownarrow$ ξ is an ordinary dist. law

- commutativity of lim's & colim's
- monoidal monads
- monoidal fibrations
- algebraic weak fact. sys.
- Abelian categories

Q: $T \circlearrowright \mathcal{C}$ (2-cat), $T^2 \overset{\mu}{\Rightarrow} T$, $\mathbb{1} \overset{\eta}{\Rightarrow} T$

Lax T-alg. $\quad A \in \mathcal{C}$ — 0-cell

$TA \xrightarrow{a} A$ — 1-cell

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\varphi\Downarrow"] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

2-cell + equations